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Motion and Uncertainty Under the Geometric-Law-of-Motion

About describing motion based on {$\underline{r}$, $\underline{v}$} phase-space vectors and a matrix format.

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	velocity			position			
	1	2	3	4	5	6	
1	0	v3i	-v2i	0	-r3i	r2i	r1
2	-v3i	0	v1i	r3i	0	-r1i	r2
3	v2i	-v1i	0	-r2i	r1i	0	r3
4	vi1	vi2	vi3	ri1	ri2	ri3	v1
5	?	?	?	?	?	?	v2
6	?	?	?	?	?	?	v3

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ABSTRACT

Bearing in mind that the *Geometric-Law-of-Motion* is not a standard concept of motion, but only a not-yet tested perspective that we have proposed in previous articles, in this article we will present it in terms of a six component *hyper-vector*[1] containing both the *position* and the *velocity* vectors. This is similar to what is frequently done when studying the evolution of a system in terms of its position (s) and its velocity (v), both along the path, where from visual phase-space diagrams can then be drawn. Such a representation, and our discussion here, will lead us to a simple matrix method for calculating the next state of the evolving system, as well as to the curious thought that perhaps it is real, and natural, to expect an intrinsic uncertainty between *position* and *velocity* in any general conditions of motion. Such, of course, makes us think of the uncertainty principle of Heisenberg, which rules the quantum-world.

1. The law of Angular-Momentum

In order to express once again what we call the *Geometric-Law-of-Motion*[2], and "deduce" it in a faster way than what was previously done in other articles, let us simply begin with the concept of force defined by Newton:

$$d(m.\underline{v})/dt = \underline{F} \qquad (1)$$

Where of course m is the mass of the body (we will assume it to be constant), $\underline{v}$ is the velocity vector, $\underline{F}$ is the force vector, and t is the time. This means that *force* is that which forces a change in the physical quantity called *linear-momentum* ($m.\underline{v}$). If there is a force acting upon the body, there must be a change in $m.\underline{v}$. Conversely, if there is a change in $m.\underline{v}$ this means there is a force acting upon the body. Knowing the forces which are present in the system one may use the above equation to, by means of integration, find the expression of the velocity ($\underline{v}$); and then, by integrating again, find how the position vector ($\underline{r}$) moves with time.

Suppose now, for the sake of simplicity, that we are sampling the reality with very small intervals of time (Δt) in which the *total force* acting upon the body can be considered a constant vector ($\underline{F}$). This allows us to perform an easy integration of Newton's equation, from which the *velocity* vector will be:

$$\underline{v}(t) = (1/m). \int \underline{F} \,.\, dt = (\underline{F}/m). \int dt$$
$$\underline{v}(t) = (\underline{F}/m).t + \underline{v}(0) \qquad (2)$$

And, by integrating this equation, we find the *position* vector:

$$\underline{r}(t) = ½ \,(\underline{F}/m).t^2 + \underline{v}(0).t + \underline{r}(0) \qquad (3)$$

[1] When a component *vector* is made of several quantities of different *physical units*, we will simply call it *hyper-vector*. The usual pair of phase-coordinates (s, v) in a dynamic system, for example, is thus a two dimensional *hyper-vector*. Similarly, the four-vector of Relativity is a four dimensional *hyper-vector*.

[2] http://paginas.fe.up.pt/%7Efeliz/e_paper32_the-geometric-law-of-motion-revised.pdf

This means that under such conditions, and once we also know the initial position $\underline{r}(0)$ and initial velocity $\underline{v}(0)$, we may completely compute both the final position $\underline{r}(t)$ and the final velocity $\underline{v}(t)$. So, the new state of the system $\{\underline{r}(t), \underline{v}(t)\}$ can be computed exactly. No uncertainty exists between these two vectors in Newtonian Physics, even if these same quantities are in effect linked by the Heisenberg uncertainty in the quantum-world.

Let us now simply make the option of multiplying both sides of the force equation of Newton by the vectorial operation ($\underline{r}$ x), again being $\underline{r}$ the position vector, and x the cross product. This leads to:

$$\underline{r} \times d(m.\underline{v})/dt = \underline{r} \times \underline{F} \quad (4)$$

Of course the right side represents *torque*, but there is some little strangeness in this equation, since the left side is not precisely the variation in time of *angular-momentum*, as it should be. This means that starting by Newton's equation we cannot directly derive the equation for the universal law of *angular-momentum*, which in effect is:

$$d(\underline{r} \times m.\underline{v})/dt = \underline{r} \times \underline{F} \quad (5)$$

Based on the same type of reasoning and using the same operation but now with the cross product substituted by the dot product, we may also write:

$$\underline{r} \,.\, d(m.\underline{v})/dt = \underline{r} \,.\, \underline{F} \quad (6)$$

Which for the same reasons must be considered inexact, and the new relation we propose is:

$$d(\underline{r} \,.\, m.\underline{v})/dt = \underline{r} \,.\, \underline{F} \quad (7)$$

What we call the *Geometric-Law-of-Motion* is the conjunction of these two equations. One is seen as the law for *angular motion*, the other is seen as the law for *radial motion*. In a compacted form we can write it as a system of two equations, linked by the hyper-vector $\{\underline{r}, \underline{v}\}$ and the vector $\underline{F}$:

$$\begin{cases} d(\underline{r} \times m.\underline{v})/dt = \underline{r} \times \underline{F} & \text{- angular law} \\ d(\underline{r} \,.\, m.\underline{v})/dt = \underline{r} \,.\, \underline{F} & \text{- radial law} \end{cases} \quad (8)$$

Finally, in an even more compact form, we can write it as a geometric-algebra law:

$$d(\underline{r} * m.\underline{v})/dt = \underline{r} * \underline{F} \quad \text{- geometric law} \quad (9)$$

2. General motion using *G-L-M*

Let us again suppose, for simplicity, that we are sampling the reality with very small intervals of time (Δt) in which the *force* ($\underline{F}$) acting upon the body can be considered constant, and let us differentiate the left side of equation (9):

$$m.[d(\underline{r})/dt] * \underline{v} + m.\, \underline{r} * d(\underline{v})/dt = \underline{r} * \underline{F} \quad (10)$$
$$[d(\underline{r})/dt] * \underline{v} + \underline{r} * d(\underline{v})/dt = \underline{r} * (\underline{F}/m) \quad (11)$$

Usually, if we blindly follow the rules of differential calculus, the term $[d(\underline{r})/dt]$ is considered equivalent to $\underline{v}$. Therefore in the *angular equation* the first term on the left naturally vanishes, leading again to equation (4). This seems to be reasonable for situations of motion where there is a slow variation in the position vector ($\underline{r}$). But, could this represent the true description of reality? On the one hand, we know that the *angular-momentum* law is written as:

$$d(\underline{L})/dt = \underline{\tau} \quad (12)$$

Being $\underline{L} = \underline{r} \times m.\underline{v}$ the *angular-momentum* vector and $\underline{\tau} = \underline{r} \times \underline{F}$ the *torque* vector. On the other hand, there are obvious variations of vectors in equation (11), therefore there are *initial* vectors and *final* vectors, and, in order to be more precise, we must distinguish all of them in that same equation. Since differential calculus does not let us do that, we transform that same equation into its equivalent using finite differences:

$$[\Delta\underline{r}/\Delta t] * \underline{v} + \underline{r} * \Delta\underline{v}/\Delta t = \underline{r} * (\underline{F}/m) \quad (13)$$
$$\Delta\underline{r} * \underline{v} + \underline{r} * \Delta\underline{v} = \Delta t \,.\, \underline{r} * (\underline{F}/m) \quad (14)$$

The question now is: are $\underline{r}$ and $\underline{v}$ in this equation *initial* vectors, or *final* vectors? By following the concept behind the differentiation of a product, both these vectors should be considered *initial* vectors, since they are treated as the constant parts of the differentiation. This means they represent the state of the system before change, which we will denote as the hyper-vector $\{\underline{r}, \underline{v}\}_i = \{\underline{r}_i, \underline{v}_i\}$. The previous equation can therefore be written as:

$$(\underline{r} - \underline{r}_i) * \underline{v}_i + \underline{r}_i * (\underline{v} - \underline{v}_i) = \Delta t \,.\, \underline{r}_i * (\underline{F}/m) \qquad (15)$$

$$\underline{r} * \underline{v}_i - \underline{r}_i * \underline{v}_i + \underline{r}_i * \underline{v} - \underline{r}_i * \underline{v}_i = \Delta t \,.\, \underline{r}_i * (\underline{F}/m) \qquad (16)$$

where we have started to use $\{\underline{r}, \underline{v}\}$ as the *final state* in order to simplify the notation. Concentrating the known terms in the right side of the equation will lead to:

$$\underline{r} * \underline{v}_i + \underline{r}_i * \underline{v} = 2.\, \underline{r}_i * \underline{v}_i + \Delta t \,.\, \underline{r}_i * (\underline{F}/m) \qquad (17)$$

And this is the general equation linking the *initial* state $\{\underline{r}_i, \underline{v}_i\}$ to the *final* state $\{\underline{r}, \underline{v}\}$ and to the force vector. Notice that in the absence of force ($\underline{F}=\underline{0}$) this equation elegantly tells us that the *final* state will be the same as the *initial* state, independent of time.

3. Computing the new-state by matrix algebra

The last equation is compact due to the usage of the geometric-algebra product. Let us now write it more explicitly:

$$\begin{cases} \underline{r} \times \underline{v}_i + \underline{r}_i \times \underline{v} = 2.\, \underline{r}_i \times \underline{v}_i + \Delta t \,.\, \underline{r}_i \times (\underline{F}/m) \\ \underline{r} \,.\, \underline{v}_i + \underline{r}_i \,.\, \underline{v} \;\; = 2.\, \underline{r}_i \,.\, \underline{v}_i + \Delta t \,.\, \underline{r}_i \,.\, (\underline{F}/m) \end{cases} \qquad (18)$$

Using the components of each vector in a coordinate system with base $(\underline{\hat{u}}_1, \underline{\hat{u}}_2, \underline{\hat{u}}_3)$, we may write the angular equation even more expanded as:

$$\begin{bmatrix} r_1 \\ r_2 \\ r_3 \end{bmatrix} \times \begin{bmatrix} v_{1i} \\ v_{2i} \\ v_{3i} \end{bmatrix} + \begin{bmatrix} r_{1i} \\ r_{2i} \\ r_{3i} \end{bmatrix} \times \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = 2.\begin{bmatrix} r_{1i} \\ r_{2i} \\ r_{3i} \end{bmatrix} \times \begin{bmatrix} v_{1i} \\ v_{2i} \\ v_{3i} \end{bmatrix} + (\Delta t/m).\begin{bmatrix} r_{1i} \\ r_{2i} \\ r_{3i} \end{bmatrix} \times \begin{bmatrix} F_1 \\ F_2 \\ F_3 \end{bmatrix} \qquad (19)$$

And, noticing that the cross product of a generic vector $\underline{r} = (r_1, r_2, r_3)$ by a generic vector $\underline{v} = (v_1, v_2\ v_3)$ is a vector that can be obtained by the matrix operation:

$$\underline{r} \times \underline{v} = \begin{bmatrix} (r_2 \,.\, v_3 - r_3 \,.\, v_2) \\ (-r_1 \,.\, v_3 + r_3 \,.\, v_1) \\ (r_1 \,.\, v_2 - r_2 \,.\, v_1) \end{bmatrix} = \begin{bmatrix} 0 & v_3 & -v_2 \\ -v_3 & 0 & v_1 \\ v_2 & -v_1 & 0 \end{bmatrix} . \begin{bmatrix} r_1 \\ r_2 \\ r_3 \end{bmatrix} \qquad (20)$$

The first member of equation (19) can simply be represented by the following matrix product over the hyper-vector $\{\underline{r}, \underline{v}\} = \{r_1, r_2, r_3, v_1, v_2\ v_3\}$:

$$\left[\begin{array}{ccc|ccc} 0 & v_{3i} & -v_{2i} & 0 & -r_{3i} & r_{2i} \\ -v_{3i} & 0 & v_{1i} & r_{3i} & 0 & -r_{1i} \\ v_{2i} & -v_{1i} & 0 & -r_{2i} & r_{1i} & 0 \end{array}\right] . \begin{bmatrix} r_1 \\ r_2 \\ r_3 \\ v_1 \\ v_2 \\ v_3 \end{bmatrix} = \underline{r} \times \underline{v}_i + \underline{r}_i \times \underline{v} \qquad (21)$$

By a similar reasoning, taking into account the radial equation:

$$\begin{bmatrix} r_1 \\ r_2 \\ r_3 \end{bmatrix} . \begin{bmatrix} v_{1i} \\ v_{2i} \\ v_{3i} \end{bmatrix} + \begin{bmatrix} r_{1i} \\ r_{2i} \\ r_{3i} \end{bmatrix} . \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = 2.\begin{bmatrix} r_{1i} \\ r_{2i} \\ r_{3i} \end{bmatrix} . \begin{bmatrix} v_{1i} \\ v_{2i} \\ v_{3i} \end{bmatrix} + (\Delta t/m).\begin{bmatrix} r_{1i} \\ r_{2i} \\ r_{3i} \end{bmatrix} . \begin{bmatrix} F_1 \\ F_2 \\ F_3 \end{bmatrix} \qquad (22)$$

It will be easy to deduce that:

$$\left[\begin{array}{ccc|ccc} v_{1i} & v_{2i} & v_{3i} & r_{1i} & r_{2i} & r_{3i} \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right] . \begin{bmatrix} r_1 \\ r_2 \\ r_3 \\ v_1 \\ v_2 \\ v_3 \end{bmatrix} = \underline{r} \,.\, \underline{v}_i + \underline{r}_i \,.\, \underline{v} \qquad (23)$$

And, joining together the angular and radial parts we can say that:

$$\left[\begin{array}{ccc|ccc} 0 & v_{3i} & -v_{2i} & 0 & -r_{3i} & r_{2i} \\ -v_{3i} & 0 & v_{1i} & r_{3i} & 0 & -r_{1i} \\ v_{2i} & -v_{1i} & 0 & -r_{2i} & r_{1i} & 0 \\ \hline v_{1i} & v_{2i} & v_{3i} & r_{1i} & r_{2i} & r_{3i} \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right] . \begin{bmatrix} r_1 \\ r_2 \\ r_3 \\ v_1 \\ v_2 \\ v_3 \end{bmatrix} = \underline{r} * \underline{v}_i + \underline{r}_i * \underline{v} \qquad (24)$$

Notice that this 6x6 matrix contains solely information of the *initial* state, therefore it will be in principle easy to build for a particular system. In effect, it is a representation of the initial *angular-momentum* ($\underline{L}_i$) and initial *action* (S_i).

We may now focus attention onto the right hand side of equation (19). Since all the parameters of this side of equation are part of the initial conditions too, and the applied force, we may simply treat it in terms of vectors resulting from cross products, that is:

$$2.\,\underline{r}_i \times \underline{v}_i = 2.\begin{pmatrix} (r_{2i}.v_{3i} - r_{3i}.v_{2i}) \\ (-r_{1i}.v_{3i} + r_{3i}.v_{1i}) \\ (r_{1i}.v_{2i} - r_{2i}.v_{1i}) \end{pmatrix} \quad (25)$$

And,

$$(\Delta t/m).\,\underline{r}_i \times \underline{F} = (\Delta t/m).\begin{pmatrix} (r_{2i}.F_{3i} - r_{3i}.F_{2i}) \\ (-r_{1i}.F_{3i} + r_{3i}.F_{1i}) \\ (r_{1i}.F_{2i} - r_{2i}.F_{1i}) \end{pmatrix} \quad (26)$$

The same reasoning applied to the equation of *radial* motion (22), however, leads to the scalars:

$$2.\,\underline{r}_i\,.\,\underline{v}_i = (r_{1i}.v_{1i} + r_{2i}.v_{2i} + r_{3i}.v_{3i}) \quad (27)$$

And,

$$(\Delta t/m).\,\underline{r}_i\,.\,\underline{F} = (\Delta t/m).(r_{1i}.F_{1i} + r_{2i}.F_{2i} + r_{3i}.F_{3i}) \quad (28)$$

These are not vectors, of course, but even so we may add them to the previous vectors in order to construct two 6 dimensions hyper-vectors, by means of which we will express the second member of equation (17), which is $2.\,\underline{r}_i * \underline{v}_i + (\Delta t/m).\underline{r}_i * \underline{F}$, as:

$$2.\begin{pmatrix} (r_{2i}.v_{3i} - r_{3i}.v_{2i}) \\ (-r_{1i}.v_{3i} + r_{3i}.v_{1i}) \\ (r_{1i}.v_{2i} - r_{2i}.v_{1i}) \\ (r_{1i}.v_{1i} + r_{2i}.v_{2i} + r_{3i}.v_{3i}) \\ 0 \\ 0 \end{pmatrix} + (\Delta t/m).\begin{pmatrix} (r_{2i}.F_{3i} - r_{3i}.F_{2i}) \\ (-r_{1i}.F_{3i} + r_{3i}.F_{1i}) \\ (r_{1i}.F_{2i} - r_{2i}.F_{1i}) \\ (r_{1i}.F_{1i} + r_{2i}.F_{2i} + r_{3i}.F_{3i}) \\ 0 \\ 0 \end{pmatrix} \quad (29)$$

Now that the details are known, we can compact again all these equations, to finally express them in terms of hyper-vectors and a matrix. We start with this last equation. Let us represent it in the form[3]:

$$2.\,\tilde{\boldsymbol{u}}_{ri,vi} + (\Delta t/m).\tilde{\boldsymbol{u}}_{ri,Fi} \quad (30)$$

Since the *final* state, which is what we want to know, will be the hyper-vector $\{\underline{r}, \underline{v}\} = \tilde{u}_{r,v}$, we may finally write the complete equation of the *Geometric-Law-of-Motion* like this:

$$[A_{ri,vi}].\,\tilde{\boldsymbol{u}}_{r,v} = 2.\,\tilde{\boldsymbol{u}}_{ri,vi} + (\Delta t/m).\tilde{\boldsymbol{u}}_{ri,Fi} \quad (31)$$

Where $[A_{ri,vi}]$ is, of course, the matrix:

$$\begin{pmatrix} 0 & v_{3i} & -v_{2i} & 0 & -r_{3i} & r_{2i} \\ -v_{3i} & 0 & v_{1i} & r_{3i} & 0 & -r_{1i} \\ v_{2i} & -v_{1i} & 0 & -r_{2i} & r_{1i} & 0 \\ v_{1i} & v_{2i} & v_{3i} & r_{1i} & r_{2i} & r_{3i} \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \quad (32)$$

The final objective is to know all the components of the final state $\tilde{\boldsymbol{u}}_{r,v}$ by means of manipulating these relations. An interesting result is the fact that if the inverse matrix of $[A_{ri,vi}]$ exists, let us denote it by $[A_{ri,vi}]^{-1}$, then the final state is directly computed by multiplying both sides of equation (31) by $[A_{ri,vi}]^{-1}$, which leads to:

$$\tilde{\boldsymbol{u}}_{r,v} = 2.[A_{ri,vi}]^{-1}.\tilde{\boldsymbol{u}}_{ri,vi} + (\Delta t/m).[A_{ri,vi}]^{-1}.\tilde{\boldsymbol{u}}_{ri,Fi} \quad (33)$$

4. Uncertainty in the final-state $\{\underline{r}, \underline{v}\}$

Since in the more general equation (31) we have a 6x6 matrix and hyper-vectors of six components, this equation would be equivalent to a system of six linear equations with six unknowns, which will be the components of the $\underline{\boldsymbol{r}}$ and $\underline{\boldsymbol{v}}$ vectors[4] for the final state $\tilde{\boldsymbol{u}}_{r,v}$. However, the two bottom rows of $[A_{ri,vi}]$ have been filled with zeros, and that results in two null equations. In effect we will have only four equations for six unknowns.

This of course means that in the most general case of motion we will not be able to precisely know the three components of both $\underline{\boldsymbol{r}}$ and $\underline{\boldsymbol{v}}$ vectors[5]. Four out of those six components will be functions of the remaining two. And this means that to specify the

[3] We will use the symbol "~" over a letter to distinguish an hyper-vector ($\tilde{\boldsymbol{v}}$) from a vector ($\underline{\boldsymbol{v}}$).

[4] If we use instead the pair $\{\underline{\boldsymbol{r}}, d\underline{\boldsymbol{r}}/dt\}$ it will result in a system of 6 differential equations on the coordinates.

[5] In Quantum Mechanics $\underline{\boldsymbol{r}}$ and $\underline{\boldsymbol{v}}$ are said to be conjugate, therefore they have associated also an uncertainty in respect to one another.

final state $\tilde{\boldsymbol{u}}_{r,v}$ we must *arbitrarily* choose two of its components. And that seems an intrinsic uncertainty in the system which naturally clouds the output of motion. Based simply on the exact knowledge of the initial position and the initial velocity vectors, and the force, we cannot always exactly predict the output of the system. We have to suppose or guess something. And such a “strange” behaviour emerges because we are studying motion based on *exact* vectors and not on vectors resulting from the abstraction of *mathematical limit* used by differential calculus. Instead of working with differentials of vectors and then integrate those differentials in order to compute the final state, we are using exact vectores for the *initial* state, and then compute the *final* state by means of an operation on such *initial* state. It seems a very natural procedure, although it gives strange results.

It is important to remember, however, that the two null rows we have introduced in matrix $[A_{ri,vi}]$ can be used as needed; for example, to contain some restrictions due to spatial or velocity constraints, or even both. Thus, in order to be more general we should consider these rows filled with some kind of 'mysterious' and generic parameters that later could be used, or not used, for describing a particular case of motion. These would represent the two hidden ('mysterious') equations missing for the exact solution to be achievable. Denoting such parameters as generic η_{ij} elements, the complete $[A_{ri,vi}]$ matrix will be written as:

$$\begin{bmatrix} 0 & v_{3i} & -v_{2i} & 0 & -r_{3i} & r_{2i} \\ -v_{3i} & 0 & v_{1i} & r_{3i} & 0 & -r_{1i} \\ v_{2i} & -v_{1i} & 0 & -r_{2i} & r_{1i} & 0 \\ v_{1i} & v_{2i} & v_{3i} & r_{1i} & r_{2i} & r_{3i} \\ \eta_{51} & \eta_{52} & \eta_{53} & \eta_{54} & \eta_{55} & \eta_{56} \\ \eta_{61} & \eta_{62} & \eta_{63} & \eta_{64} & \eta_{65} & \eta_{66} \end{bmatrix} = [A_{ri,vi}] \qquad (34)$$

5. What about the opposite?

The discussion till now was based on the option of considering $\underline{r}$ and $\underline{v}$ vectors in the equation of G-L-M (14) as *initial* vectors. Differences of *velocity* and of *position* were taken in respect to these *initial* vectors. When we use differential calculus, looking at this equation “from the left” or “from the right” leads us to the same result, since the idea of “evolution” is forced to be “compacted” into an *instantaneous* vector, which is acting in the system at precisely the same time as the others. When using differences, however, we have to decide which are the initial and final vectors in each difference. And as such, as we will see, in this case leads to completely different results. Starting again with equation (14):

$$\Delta\underline{r} * \underline{v} + \underline{r} * \Delta\underline{v} = \Delta t \,.\, \underline{r} * (\underline{F}/m) \qquad (14)$$

If we now consider $\underline{v}$ and $\underline{r}$ to be final vectors, we will have:

$$(\underline{r} - \underline{r}_i) * \underline{v} + \underline{r} * (\underline{v} - \underline{v}_i) = \Delta t \,.\, \underline{r} * (\underline{F}/m) \qquad (35)$$

Therefore,

$$\underline{r} * \underline{v} - \underline{r}_i * \underline{v} + \underline{r} * \underline{v} - \underline{r} * \underline{v}_i += \Delta t \,.\, \underline{r} * (\underline{F}/m) \qquad (36)$$

$$2.\underline{r} * \underline{v} - \underline{r}_i * \underline{v} - \underline{r} * \underline{v}_i = \Delta t \,.\, \underline{r} * (\underline{F}/m) \qquad (37)$$

$$(2.\underline{r} - \underline{r}_i) * \underline{v} - \underline{r} * (\underline{v}_i + \Delta t \,.(\underline{F}/m)) = 0 \qquad (38)$$

This would be the new equation responsible for controlling motion. However, now it seems not so easy to compute the final state $\{\underline{r}, \underline{v}\}$ by means of an operation over the initial state $\{\underline{r}_i, \underline{v}_i\}$, and, at the same time, a null force ($\underline{F}=\underline{0}$) would lead to the very strange result:

$$2.\underline{r} * \underline{v} - \underline{r}_i * \underline{v} = \underline{r} * \underline{v}_i \qquad ??? \qquad (39)$$

6. Cross terms in motion, and the projectile case

Due to the fact that G-L-M is a “law” based on product operations between *position* vectors and *velocity* vectors, that is, based on *areal velocities* ($\underline{r} \times \underline{v}$) and *actions* ($\underline{r} \,.\, \underline{v}$), which in many cases are thought to be constants of motion, we must expect the set of equations generated by G-L-M to appear with some sort of cross dependency between the components of $\underline{r}$ and the components of $\underline{v}$, that is, these vectors are *entwined* with each other. This of course may lead to complex systems of equations which should be better computed by numerical methods or simulated by computer programming. Could such a complex dependency be responsible for the curious type of orbitals that exist in the atomic-world? Well, as a practical example, we will try to represent here the “simple” case of the projectile motion under gravity, as depicted in the next figure.

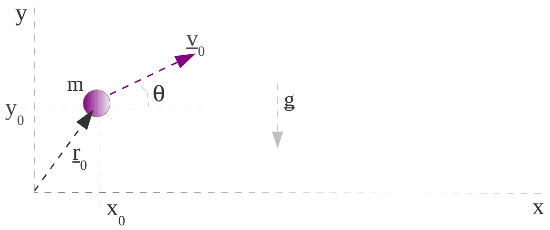

Fig. 1 Two dimensional projectile motion, being $\underline{r}_0$ the initial position vector, $\underline{v}_0$ the initial velocity vector, $\underline{g}$ the acceleration of gravity, and θ the initial angle with respect to the horizontal. No friction is considered.

The $[A_{ri,vi}]$ matrix is given by:

$$\begin{bmatrix} 0 & 0 & -v_{02} & 0 & 0 & y_0 \\ 0 & 0 & v_{01} & 0 & 0 & -x_0 \\ v_{02} & -v_{01} & 0 & -y_0 & x_0 & 0 \\ v_{01} & v_{02} & 0 & x_0 & y_0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \qquad (40)$$

Where $v_{01} = v_0.cos(\theta)$ and $v_{02} = v_0.sin(\theta)$. At the same time, from equation (29), the hyper-vector $\tilde{\boldsymbol{u}}_{ri,vi}$ contains the components of the initial angular momentum and the initial action in respect to the observer. That is:

$$\tilde{\boldsymbol{u}}_{ri,vi} = \begin{bmatrix} 0 \\ 0 \\ x_0.v_{02} - y_0.v_{01} \\ (x_0.v_{01} + y_0.v_{02}) \\ 0 \\ 0 \end{bmatrix} \qquad (41)$$

And the hyper-vector $\tilde{\boldsymbol{u}}_{ri,Fi}$, related to the energies involved in the system, may be written as:

$$\tilde{\boldsymbol{u}}_{ri,Fi} = \begin{bmatrix} 0 \\ 0 \\ x_0.F_{02} - y_0.F_{01} \\ (x_0.F_{01} + y_0.F_{02}) \\ 0 \\ 0 \end{bmatrix} \qquad (42)$$

Using this information in the G-L-M expression,

$$[A_{ri,vi}].\ \tilde{\boldsymbol{u}}_{r,v} = 2.\ \tilde{\boldsymbol{u}}_{ri,vi} + (\Delta t/m).\tilde{\boldsymbol{u}}_{ri,Fi} \qquad (31)$$

will lead to the following set of equations, with the unknowns represented in red:

$$\begin{cases} -v_{02}.z + y_0.v_z = 0 & \Rightarrow \quad v_z = (v_{02}/y_0).z \\ v_{01}.z - x_0.v_z = 0 & \Rightarrow \quad v_z = (v_{01}/x_0).z \\ v_{02}.x - v_{01}.y - y_0.v_x + x_0.v_y = 2.(x_0.v_{02} - y_0.v_{01}) + (x_0.F_{02} - y_0.F_{01}).(t/m) \\ v_{01}.x + v_{02}.y + x_0.v_x + y_0.v_y = 2.(x_0.v_{01} + y_0.v_{02}) + (x_0.F_{01} + y_0.F_{02}).(t/m) \\ 0 = 0 \\ 0 = 0 \end{cases} \qquad (43)$$

The work of *inertial forces* does not have to be included in the energy present in the system, as it is indirectly included in the angular momentum. Thus, in this case, the total force vector is:

$$\underline{F} = (F_{01}, F_{02}) = (0, -m.g) \qquad (44)$$

So, we may now leave to the reader the calculation of this system of equations in $\underline{r}$ and $\underline{v}$, or, even more challenging, the system of differential equations in $\underline{r}$ and $d\underline{r}/dt$. Notice, however, that these equations can also be analysed in the phase-space $\{\underline{r}, \underline{v}\}$ since each component of $\underline{v}$ will be a function of the components of $\underline{r}$ and of t. That is, in general we will have:

$$v_x = dx/dt = f_1(x, y, t)$$
$$v_y = dy/dt = f_2(x, y, t) \qquad (45)$$

For a fast and simplified glimpse on this case, however, let us consider all the parameters equal to 1, except the force. The two relevant equations from system (43) will then be reduced to:

$$\begin{cases} x - y - v_x + v_y = (F_{02} - F_{01}).t \\ x + y + v_x + v_y = 4 + (F_{01} + F_{02}).t \end{cases} \qquad (46)$$

$$\begin{cases} v_y = v_x + y - x + (F_{02} - F_{01}).t \\ v_x = -v_y - y - x + 4 + (F_{01} + F_{02}).t \end{cases} \qquad (47)$$

$$\begin{cases} v_y = v_x + y - x + (F_{02} - F_{01}).t \\ v_x = -v_x - y + x - (F_{02} - F_{01}).t - y - x + 4 + (F_{01} + F_{02}).t \end{cases} \qquad (48)$$

$$\begin{cases} v_y = v_x + y - x + (F_{02} - F_{01}).t \\ v_x = -v_x - 2.y + 4 + t.(F_{01} + F_{02} + F_{01} - F_{02}) \end{cases} \qquad (49)$$

$$\begin{cases} v_y = -y + 2 + t.F_{01} + y - x + t.(F_{02} - F_{01}) \\ v_x = -y + 2 + t.F_{01} \end{cases} \qquad (50)$$

$$\begin{cases} v_y = -x + 2 + t.F_{02} \\ v_x = -y + 2 + t.F_{01} \end{cases} \quad (51)$$

Since $F_{01} = 0$, we will finally have:

$$\begin{cases} v_y = -x + 2 - m.g.t \\ v_x = -y + 2 \end{cases} \quad (52)$$

We can see that these expressions describe a basic behaviour similar to what we were expecting, but they also include a cross term on the other coordinate, that is, $v_y = dy/dx$ is also dependent on x and $v_x = dx/dt$ is dependent on y. In a certain way this is not difficult to interpret, since now the base law of motion also includes the time variation of the position vector, and not only the time variation of the velocity, as proposed by Newton, and this simple fact results in an indirect inclusion of Euler and Coriolis accelerations into the equations of motion. These effects seem to be related to the overall process of conservation of *angular-momentum* and the preservation of the state of motion. Since even in free motion, along a straight-line, the *angular-momentum* with respect to any observer is conserved, it also is very probable that *inertia* itself can be seen as an indirect result of the conservation of *angular-momentum* by means of an *Euler/Coriolis* mechanism acting continuously upon the body. That is, *inertia* would be a kind of gyroscopic effect.

7. Conclusions

Although leading to some unusual equations of motion, with cross dependency between *position* and *velocity* (these parameters becoming *conjugate*), we have shown here a way of writing and using the *Geometric-Law-of-Motion* to represent motion as a single equation of *hyper-vectors*, driven by a 6x6 matrix of the initial state of motion, and probably spatial and speed restrictions. Would this be precise enough to be used in practice, and we would be in the presence of a compact description of motion based on a single equation containing both linear and angular effects. The challenge, however, is that most of the times this representation leads to a system of equations where the r_j and v_j components along a certain coordinate may also be dependent on the r_i and v_i components along other coordinates, making it difficult to visualise the evolution of the system in terms of analytic functions; therefore this system should better be solved and studied by means of computational methods. Or by means of slices on the $\{\underline{r}, \underline{v}\}$ phase-space. Although apparently strange, the curious *entanglement* detected between *position* and *velocity* make us also wonder if the uncertainty principle ruling the atomic-world could simply be a stronger expression of that what happens in our world, due to the extremely high velocities and small distances in which atomic particles move. There, the *Euler/Coriolis* effects are expected to be very strong.

We would be delighted to receive some feedback from anyone who meanwhile will test these concepts either theoretical or in practice.

Author's Biography:

J. Manuel Feliz-Teixeira, graduated in Physics, Masters in Mechanical Engineering and Doctorate in Sciences of Engineering from the University of Porto. He has been dedicated to various fields of knowledge and several industrial projects. More recently, he became mainly interested in lecturing Physics and studying *electromagnetism* and *gravitation* by means of rethinking the classical principles.