Tutorial III
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Decision-Making Methods

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Outline of the tutorial

- Basic ideas and concepts
- Modeling
- Identifying efficient alternatives (multiattribute problems)
- Generating efficient alternatives (multiobjective problems)
- Methods for deterministic problems
- Decisions under uncertainty
  - Applying decision paradigms
  - Using multiple (risk) indices
  - Some ideas about fuzzy modeling
- Final remarks

Basic ideas and concepts

Trivial decision problems

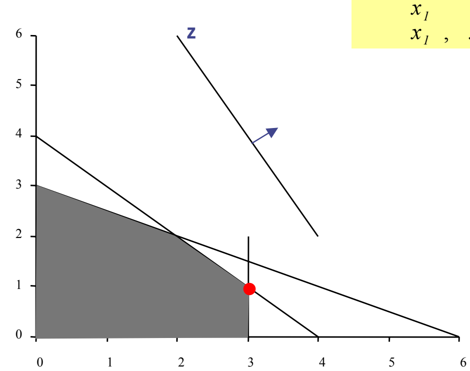
- Minimize Cost

n	Cost
1	65
2	58
3	72
4	72
5	60
6	65
7	71
8	51
9	67
10	90
11	67
12	86
13	66
14	52
15	76

- Maximize profit z

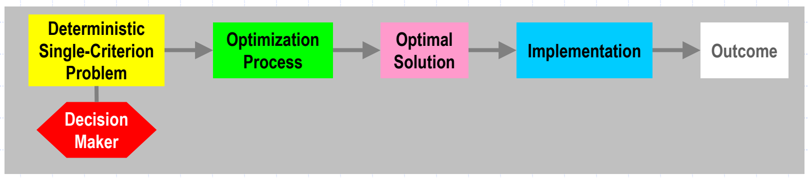
$$\max \ z = 2x_1 + x_2$$

$$\begin{aligned} suj:\quad x_1 + x_2 &\leq 4 \\ x_1 + 2x_2 &\leq 6 \\ x_1 &\leq 3 \\ x_1 \ , \ x_2 &\geq 0 \end{aligned}$$



Deterministic, single-criterion problems

- The role of the decision maker
  - The DM participates only in the problem formulation
  - The rest of the process is mainly technical, leading (hopefully) to the optimal solution
  - The decision is embedded in the problem formulation



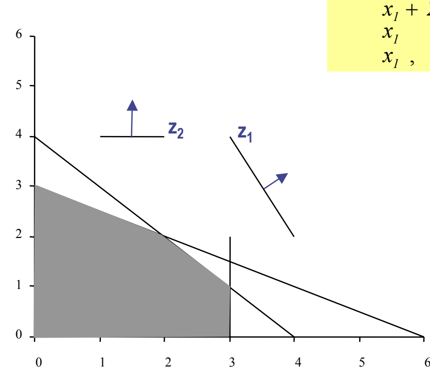
Deterministic multicriteria problems

- Minimize Cost
- Maximize Reliability

n	Cost	Reliability
1	65	0.994586
2	58	0.993677
3	72	0.995333
4	72	0.995531
5	60	0.994064
6	65	0.994641
7	71	0.995954
8	51	0.992906
9	67	0.995111
10	90	0.998551
11	67	0.995425
12	86	0.997641
13	66	0.994653
14	52	0.992848
15	76	0.995913

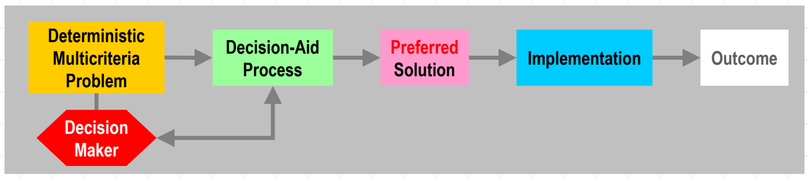
- Maximize profit z_1
- Maximize export z_2

$$\max\ z_1 = 2x_1 + x_2$$
$$\max\ z_2 = x_2$$
$$suj:\ \begin{aligned} x_1 + x_2 &\le 4 \\ x_1 + 2x_2 &\le 6 \\ x_1 &\le 3 \\ x_1,\ x_2 &\ge 0 \end{aligned}$$



Deterministic, multicriteria problems

- The role of the decision maker
  - The DM participates in the problem formulation
  - The structure of preferences of the DM must be incorporated in the problem
  - The process leads to the preferred solution



- Important message:
  - There is no way to "solve" a MC problem without incorporating the DM's preferences

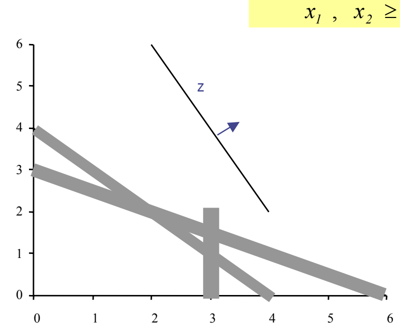
Different types of uncertainty

- Probabilistic - Different scenarios with probabilities

n	Cost		
	C1 (0.1)	C2 (0.6)	C3 (0.3)
1	59	65	75
2	50	58	71
3	68	72	60
4	69	72	62
5	53	60	63
6	51	59	65
7	68	71	77
8	56	57	75
9	62	58	80
10	62	55	70

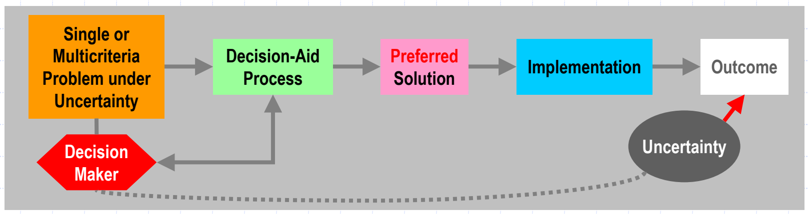
- Fuzzy - Vague or imprecise constraints

$$\max\ z = 2x_1 + x_2$$
$$suj:\quad x_1 + x_2 \ \tilde{\le}\ 4$$
$$x_1 + 2x_2 \ \tilde{\le}\ 6$$
$$x_1 \ \tilde{\le}\ 3$$
$$x_1\ ,\ x_2 \ge 0$$



Problems under uncertainty

- The role of the decision maker
  - The DM participates in the problem formulation and in the uncertainty characterization
  - The preferred solution results from the incorporation in the problem of the structure of preferences of the DM, including his risk attitude



- Important message:
  - There is no way to "solve" a problem under uncertainty without incorporating the DM's preferences and risk attitude

Modeling

Modeling

- Identification of
  - Alternatives
    - Or the constraints that define them implicitly
  - Relevant criteria (how to compare the outcomes of two alternatives)
  - Main sources of uncertainty
- Formulation of
  - Decision variables
  - External variables and parameters
  - Coherent family of **criteria**
  - Attributes
    - How to measure the satisfaction in each criterion
    - (e.g. **Criterion** – Minimize environmental impact. **Attribute** - %CO_2)

Modeling

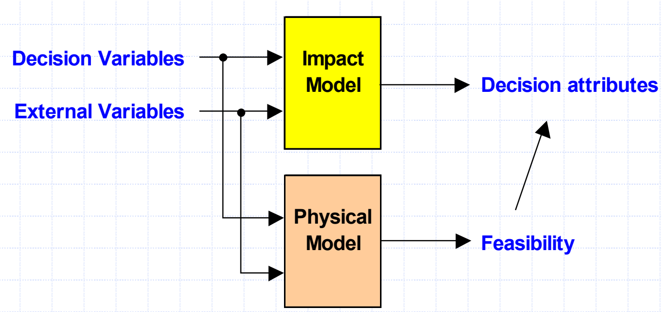
- A coherent family of criteria must be:
  - Exhaustive – All important points of view must be included
  - Consistent – If two alternatives A and B are equivalent except in criterion k, and A_k is better than B_k, then A must be at least as good as B
  - Non-redundant - Eliminating a criterion leads to the violation of one of the preceding conditions
- Other desirable proprieties
  - Legibility - The number of criteria used must be relatively low
  - Operationality - The family of criteria must be accepted by the stakeholders and the decision makers

Modeling

- Impact
  - Outcome of each particular decision (e.g. objective functions)
- Physical model
  - How to evaluate feasibility (e.g. mathematical constraints)
- Uncertainty
  - Probability distributions
  - Scenarios (with or without probabilities)
  - Possibility distributions (fuzzy sets)

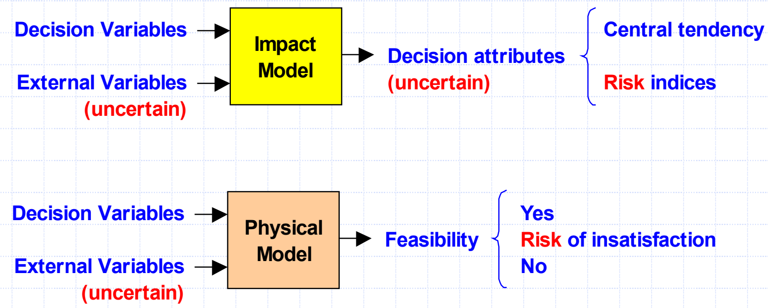
Modeling

- No uncertainty



Modeling

- Uncertain environment



Some definitions

- Dominated (inferior) alternative
  - A solution is dominated iff there exists another one that is better in at least one criterion, without being worse in any of the remaining criteria
- Efficient (nondominated, noninferior, Pareto optimal) alternative
  - A solution is efficient iff it is not dominated by any other feasible alternative
- Ideal
  - (Non feasible) solution that joins up the individual optima
  - Defined only in the attributes' space

Identifying efficient alternatives (multiattribute problems)

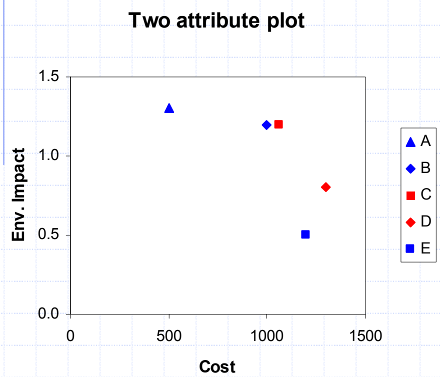
Multiattribute problems

Alternatives	C_1	C_2	*Criteria*	C_m
A_1	a_{11}	a_{12}	$\cdots$	a_{1m}
A_2	a_{21}	a_{22}	$\cdots$	a_{2m}
$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$
A_n	a_{n1}	a_{n2}	$\cdots$	a_{nm}

Attributes
may be
real numbers, intervals,
probability distributions,
possibility distributions,
qualitative labels

- Main characteristics
  - The alternatives are completely defined and assumed feasible
  - Attributes may be determinist, probabilistic, fuzzy (or mixed)
  - The problem may be:
    - Choice – Select the best alternative
    - Ranking – Draw a complete order of the alternatives
    - Sorting – Select the best k alternatives from a list of n>k

Example



- E dominates D
  - **E** is strictly better than **D** in both criteria
- B dominates C
  - **B** is strictly better than **C** in the **Cost** criterion
  - **B** is not worse than **C** in any criterion
- C and D are dominated
- A, B and E are efficient
  - They are not dominated by any other alternative

NB:
A possible rank: B, C, E, D, A

Examples

- Minimize Cost
- Maximize Reliability

n	Cost	Reliability
1	65	0.994586
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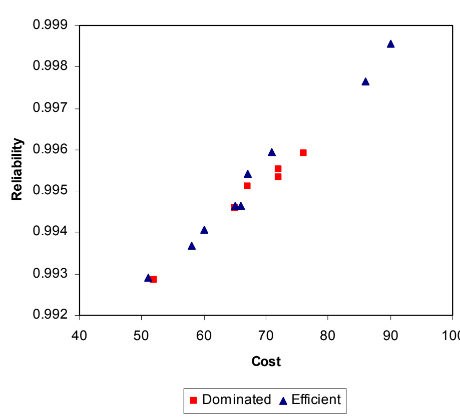


Generating efficient alternatives (multiobjective problems)

Multiobjective problems

$$\min \quad \mathbf{f}(\mathbf{x}) = \begin{cases} f_1(\mathbf{x}) \\ f_2(\mathbf{x}) \\ \dots \\ f_m(\mathbf{x}) \end{cases}$$

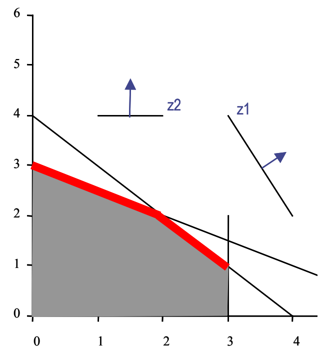
$$st: \quad \mathbf{g}(\mathbf{x}) = \mathbf{0}, \quad \mathbf{h}(\mathbf{x}) \leq \mathbf{0}, \quad \mathbf{x} \geq \mathbf{0}$$

$\mathbf{x}$	vector of decision variables *(may include integer or binary variables)*
$\mathbf{f}(\mathbf{x})$	vector of objective functions
$\mathbf{g}(\mathbf{x})$	set of equality constraints
$\mathbf{h}(\mathbf{x})$	set of inequality constraints

- Main characteristics
  - Alternatives are not known in advance
  - Optimization procedures are always needed
  - May have a big number of constraints and decision variables
  - May not be completely described by the mathematical formulation
  - Sometimes interpreted as optimization problems with more than one objective function (vector optimization)

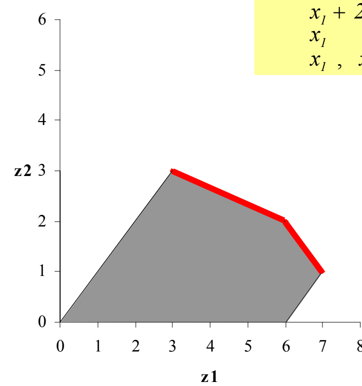
Decision space vs attribute's space

- Decision space



- Attribute's space

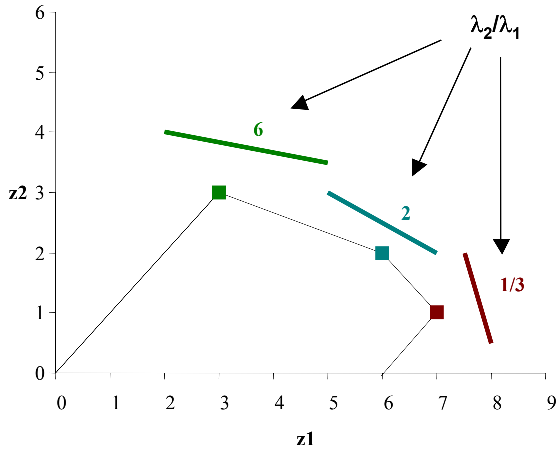
$$\begin{aligned}
\max\ & z_1 = 2x_1 + x_2 \\
\max\ & z_2 = x_2 \\
suj:\ & x_1 + x_2 \le 4 \\
& x_1 + 2x_2 \le 6 \\
& x_1 \le 3 \\
& x_1\ ,\ x_2 \ge 0
\end{aligned}$$



Generation methods

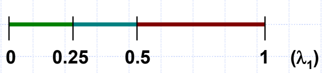
- Parametric variation of $\lambda>0$ in $min\ f(\mathbf{x}) = \sum_{i=1}^{m} \lambda_i f_i(\mathbf{x})$
  - The optimal solution of this auxiliary problem is an efficient solution of the original multiobjective problem
  - The parameters λ are only instrumental (not judgments of the DM)
- Constrained optimization
  - Define additional constraints in n-1 objective functions
  - Optimize the remaining objective function
  - Repeat for different RHS values of the additional constraints
- Multiobjective simplex
- Multiobjective metaheuristics

Parametric variation

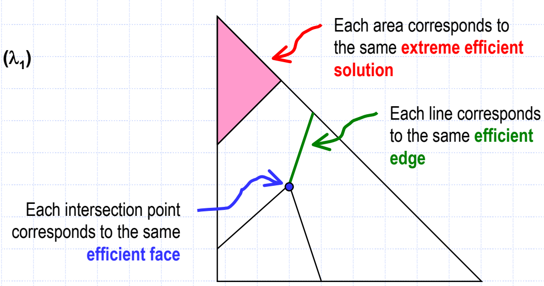


Parametric variation

- In MO linear problems, post-optimization (parametric analysis) can be used to find all the efficient solutions
  - e.g. (previous problem)

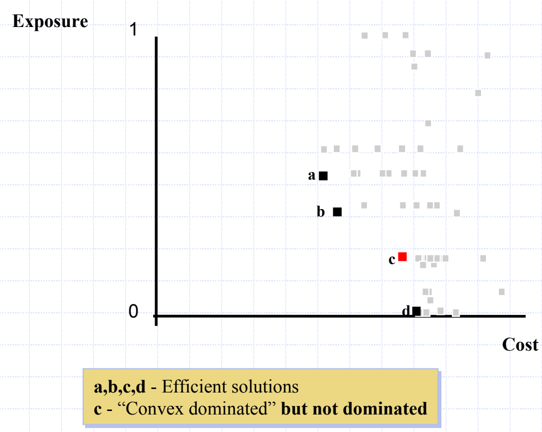


e.g. (tricriteria problem)

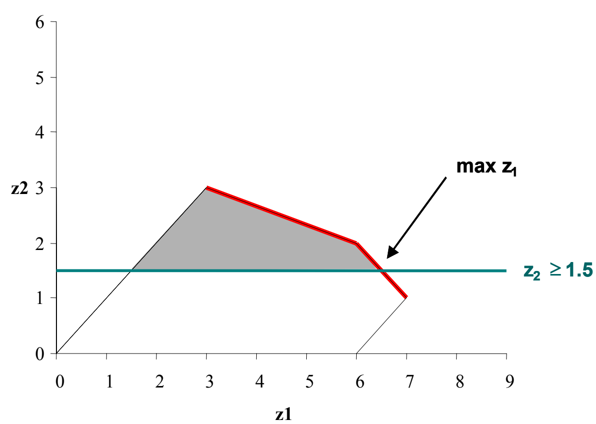


Parametric variation

- Difficulties in combinatory problems
  - Some efficient solutions are never selected



Constrained optimization (ε - constraint)



Strategies for MO problems

- Generation methods
- Aggregation of criteria (use of a value function)
  - Transforms the problem into an optimization one
- Interactive methods
  - Based on an implicit value function (never explicitly known!)
    - Geoffrion-Dyer-Feinberg, Surrogate Worth Trade-off, Zionts-Wallenius
  - Without special conditions
    - STEM, Trimap
- Goal programming

Some arguments

Strategy	Pro	Con
Generation	Doesn't have parameters Gives the global picture Doesn't require the DM's presence	Doesn't produce a solution or an order Risk of generating to many solutions Heavy calculations
Aggregation	Leads to optimization Induces a total order No further intervention of the DM	Difficulties in building the Value Function Some arbitrariness Tendency to predefinitions and confusion between OF and VF
Interactive	Reduces information overload Easier calculations (in general) Induces learning	Loss of holistic vision Produces only a final solution May need many judgments
Goal Prog.	Well established in OR Easy to apply Adequate to large dimension problems	Only linear problems Needs goal definition Requires a lexicographic order of the criteria (no compensation)

Aggregation (use of a value function)

$$\min \quad \mathbf{f}(\mathbf{x}) = \begin{cases} f_1(\mathbf{x}) \\ f_2(\mathbf{x}) \\ \cdots \\ f_m(\mathbf{x}) \end{cases}$$

$$st: \quad \mathbf{g}(\mathbf{x}) = \mathbf{0}, \quad \mathbf{h}(\mathbf{x}) \le \mathbf{0}, \quad \mathbf{x} \ge \mathbf{0}$$

→

$$v(\mathbf{x}) = v[f_1(\mathbf{x}), f_2(\mathbf{x}), \dots, f_m(\mathbf{x})]$$

↓

$$\max \quad v(\mathbf{x})$$

$$st: \quad \mathbf{g}(\mathbf{x}) = \mathbf{0}, \quad \mathbf{h}(\mathbf{x}) \le \mathbf{0}, \quad \mathbf{x} \ge \mathbf{0}$$

$\mathbf{x}$ vector of decision variables
(may include integer or binary variables)
$\mathbf{f}(\mathbf{x})$ vector of objective functions
$\mathbf{g}(\mathbf{x})$ set of equality constraints
$\mathbf{h}(\mathbf{x})$ set of inequality constraints

Interactive approaches

(typically, only for MO linear problems)

- General procedure
  1. Find an initial solution (efficient)
  2. Ask the DM if he is satisfied → if he is, this is the preferred solution. STOP
  3. Ask the DM which criteria he wants to improve and which criteria he accepts to worsen
  4. Use the precedent information to find a new solution
  5. Return to 2

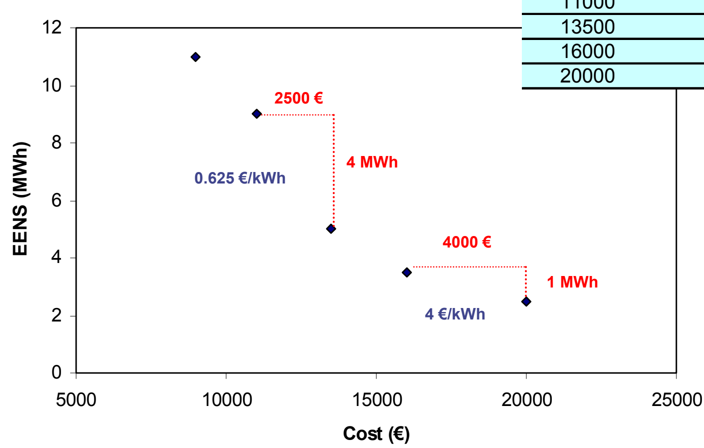
- Some classics
  - STEM
    - STRANGE
  - Zionts-Wallenius
  - Interval Criterion Weights
  - Surrogate Worth Trade-off
  - Geoffrion-Dyer-Feinberg
  - Pareto Race
  - Trimap

Methods for deterministic problems

Trade-off analysis

- 5 possible investment plans

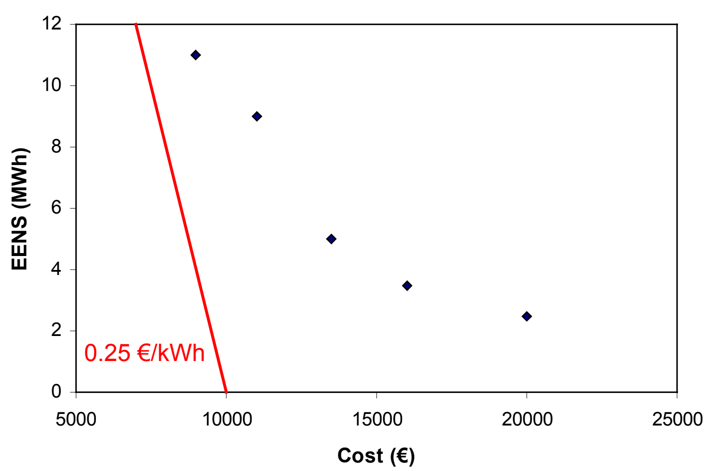
Cost (€)	EENS (MWh)
9000	11
11000	9
13500	5
16000	3.5
20000	2.5



Trade-off analysis

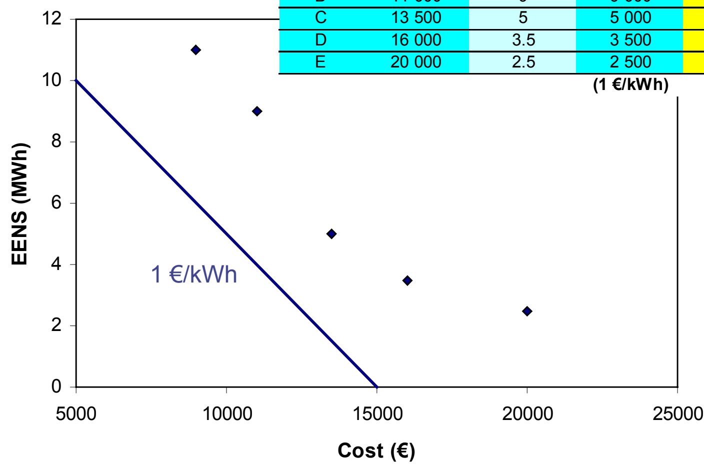
- When comparing B to A (two efficient alternatives)
  - We gain something in one criterion
  - We loose something in another criterion
- If we have a reference value for the trade-off
  - We know immediately if we prefer A or B
  - It's easy to select the preferred alternative

Trade-off analysis

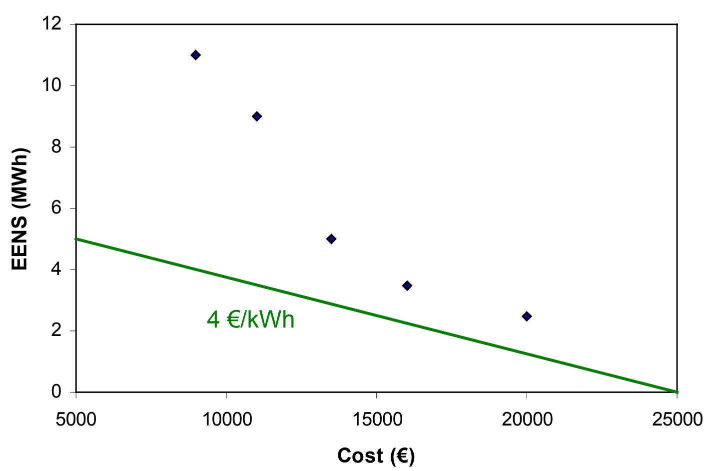


Trade-off analysis

alternative	Cost (€)	EENS (MWh)	EENS (€)	Total
A	9 000	11	11 000	20 000
B	11 000	9	9 000	20 000
C	13 500	5	5 000	**18 500**
D	16 000	3.5	3 500	19 500
E	20 000	2.5	2 500	22 500
			(1 €/kWh)	

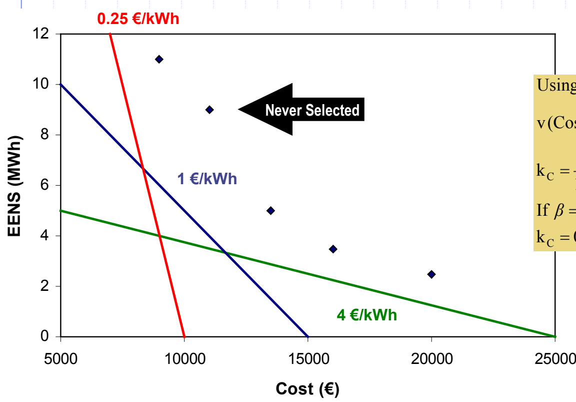


Trade-off analysis



Trade-off analysis

- Each trade-off β defines a family of indifference lines
  - f(Cost, EENS) = Cost + β. EENS β in €/MWh



Using a normalized **value function**:

$$v(\text{Cost}, \text{EENS}) = k_C \frac{20000 - \text{Cost}}{20000 - 9000} + k_E \frac{11 - \text{EENS}}{11 - 2.5}$$

$$k_C = \frac{11000}{11000 + 8.5\beta} \qquad k_E = \frac{8.5\beta}{11000 + 8.5\beta}$$

If $\beta = \mathbf{1000}$ **€/MWh**

$k_C = 0.564 \quad k_E = 0.436$

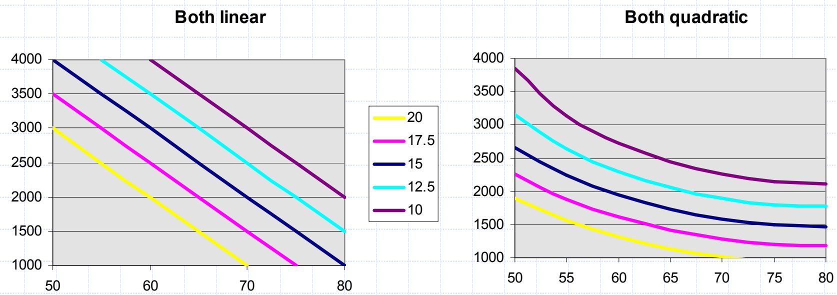
Trade-off analysis

- Conclusions:
  - Constant trade-offs lead to linear indifference curves
  - … and to linear value functions
  - … with constant weights
    - that have no special meaning as indicators of the relative importance of the criteria in general

- Important issues
  - The process may be extended to more than two criteria
  - Trade-offs are not always constant
    - e.g. beyond a certain level, your willingness to pay for extra reliability decreases
  - … leading to non-linear indifference curves
  - … and non-linear value functions
    - but generally still additive, with constant parameters

Indifference curves

- Indifference curves join all the points with the same *global value*
- The DM is indifferent between two points in the same curve



Minimization in both criteria
Value scale (20≻17.5, etc)

Summarizing

- Indifference curve (attribute space)
  - Set of the alternatives that are valued the same way by the Decision Maker
  - The indifference curves completely describe the structure of preferences of the Decision Maker
- Trade-off between two attributes X and Y
  - What you must lose in X to increase one unit in Y, while staying in the same indifference curve (slope of the curve)
- Weights
  - If and only if the trade-offs are constant, weights are constant

Value functions

- A formal way to address decision problems
  - Sometimes called deterministic utility functions
- If some conditions are met, there exists a real value function v() such that:
  - $A \succ B \Leftrightarrow v(A) > v(B)$
  - $A \sim B \Leftrightarrow v(A) = v(B)$
- Use of an additive value function requires:
  - Verifying assumptions
  - Construction of the individual value functions
  - Indifference judgments to build the multiattribute value function
    - **No** naïve weights asked directly to the DM**!**

Individual value functions

- Individual (or conditional) value function
  - Measures the satisfaction in one criterion, regardless of the values of the other criteria
- Typical value functions (minimization):
  - Linear $v(x) = x_N = \frac{x_{max} - x}{x_{max} - x_{min}}$
  - Quadratic 1 $v(x) = (x_N)^2$
  - Quadratic 2 $v(x) = 2.x_N - (x_N)^2$
  - Exponential $v(x) = \frac{e^{a.x_N} - 1}{e^a - 1}$

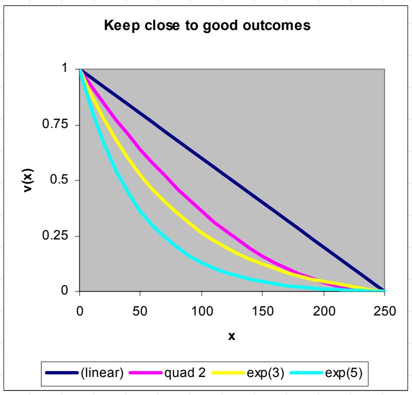
Generally **v(x)** is normalized, with:

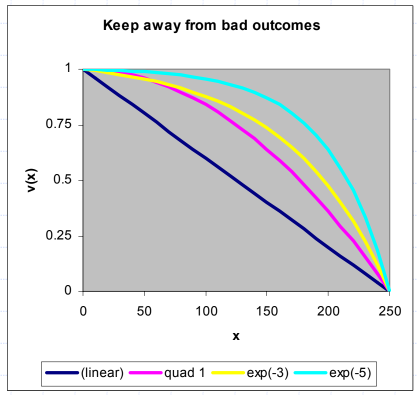
v(best x) = 1

v(worst x) = 0

Individual value functions

- The convexity of the value function reflects the variation of the DM's satisfaction in the range of the attribute
  - The same difference in the attribute (e.g. 50-100 and 200-250) does not correspond to the same increase in satisfaction (exception: linear v.f.)





MA value function - parameters

- Assess the parameters k_1 and k_2 $v(A) = k_1 v_1(A_1) + k_2 v_2(A_2)$
  - Build "extreme" alternatives:

$Ideal : best\ A_1, best\ A_2$	$P : best\ A_1, worst\ A_2$	$Q : worst\ A_1, best\ A_2$
$v = 1, v_1 = 1, v_2 = 1$	$v_1 = 1, v_2 = 0$	$v_1 = 0, v_2 = 1$
$k_1 + k_2 = 1$	$v(P) = k_1$	$v(Q) = k_2$

  - Ask for a judgment (eg: P $\succeq$ Q, that implies $k_1 \geq k_2$)
  - Find M = (z, worst A_2) ~ Q
    - Then:

$$v(M) = v(Q) \Rightarrow k_1 v_1(z) = k_2$$

$$k_1 = \frac{1}{1 + v_1(z)} \qquad k_2 = 1 - k_1$$

    - **This is very different from asking directly for weights!**

Example

Cost (€)	EENS (MWh)
9000	11
11000	9
13500	5
16000	3.5
20000	2.5

$$v(\text{Cost},\text{EENS}) = k_C \frac{20000-\text{Cost}}{20000-9000} + k_E \frac{11-\text{EENS}}{11-2.5}$$

- Build "extreme" alternatives:
  - P=(9000, 11), Q=(20000, 2.5)
- Search for an indifference
  - *P or Q?*
    - The DM says P $\succ$ Q
  - *P'=(11000, 11) or Q?*
    - P' $\succ$ Q
  - *P"=(12000, 11) or Q?*
    - Q $\succ$ P"
  - M=(11500, 11) ~ Q=(20000, 2.5)
  - $v_C(11500)=0.773$
  - $k_C=0.564$ $k_E=0.436$

NB:
8 500 € compensates 8.5 MWh
Trade-off = 1 €/kWh

$$v(M) = v(Q) \Rightarrow k_C v_C(z) = k_E$$

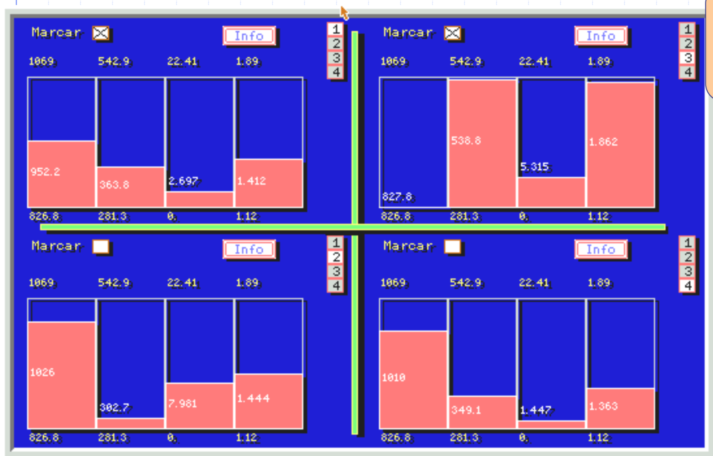
$$k_C = \frac{1}{1+v_C(z)} \qquad k_E = 1 - k_C$$

Other methodologies for MA problems

- AHP - Analytic Hierarchy Process
  - Hierarchical organization of the criteria
  - Based on comparison matrices of binary comparisons
    - Between sub-criteria, regarding the parent criterion
    - Between alternatives, regarding all the last level criteria
    - Inconsistencies are allowed (to a certain degree)
- Decision-aid methodologies *(the French School)*
  - Preference models include indifference and veto thresholds, weak and strong preference, and incomparability situations
  - Aggregation of preferences mainly by rules (as opposed to formulas)
  - Methods - ELECTRE I, IS, II, III, IV, Tri, PROMETHEE, GAIA
- Non-parametric methodologies
  - Based on successive choices between a reduced number of alternatives

SAM – a cluster based methodology

- an example in distribution planning (extract)



(minimize)
A - Investment Cost
B - EPNS
C - Voltage Quality
D - Power Losses

53 original alternatives

4 clusters

The DM chooses clusters 1 and 3

Decisions under uncertainty

Uncertainty issues

- Uncertainty about the data
  - Loads, costs, wind power, hydro inflows, economical parameters
  - Reliability parameters
- Uncertainty about the outcome of random variables
  - Quality of service indices
- Uncertainty about the behavior of other agents
  - Regulatory decisions
  - Environmental pressure
  - Competitors' decisions
- Uncertainty about the model

Uncertainty models

- Scenarios
  - with or without probabilities
- Probabilistic models
  - continuous, discrete, *subjective*
- Fuzzy models
  - (intervals are fuzzy sets)

Scenarios

- Scenarios are coherent estimates of the uncertain environment
  - Taking into account the relations and dependencies between variables
  - Although quantitatively characterized, they correspond to qualitatively different realizations of the uncertainties
    - e.g. "Moderate economic development", "Economic stagnation"
  - Probabilities may be assigned to each scenario
    - Sometimes, *subjective probabilities*
    - But also *interval* or *fuzzy* probabilities

- Output
  - Impact of the decisions in each scenario

Probabilistic models

- Input
  - Continuous variables with known distributions
  - Discrete independent variables
  - Scenarios

- Output
  - Probability distributions of the attributes
  - Expected values of the attributes
  - Other moments of the distributions of the attributes

- Methods
  - Analytic
  - Simulation (e.g. Monte-Carlo)

Fuzzy models

- Basic concept
  - The degree of membership of an element of the universe of discourse to the concept associated to the fuzzy set may take any value in (0,1)
    - e.g. **u(7, "near 9")=0.3**
    - e.g. **u(17 min, "a quarter of hour")=0.9**
- Typical applications
  - "This load will be around 800 kW"
  - "The consumption will grow from 3 to 5%"
  - "The deficit should not exceed too much 3%"
- Output
  - Possibility distributions of the attributes
  - Robustness regarding constraint violations

A global view on alternatives (one criterion)

- In each criterion, each alternative has an outcome:
  - a real number, when no uncertainty exists
  - a list of real numbers, when a finite number of scenarios exists
  - a list of pairs (attribute value, probability), when a finite number of scenarios exists with assigned probabilities
  - a discrete probability distribution or a probability density function, when the uncertainty is probabilistic
    - Dependencies may exist between random variables
  - a possibility distribution, when a fuzzy model is used
    - e.g. to describe mathematically qualitative labels
    - Intervals are a particular case of fuzzy sets
- An alternative may be a stream of decisions over time
  - Including conditional decisions in the future (strategy)
- Hedging policies can "generate" additional alternatives

Uncertain environment (probabilistic)

- Single criterion

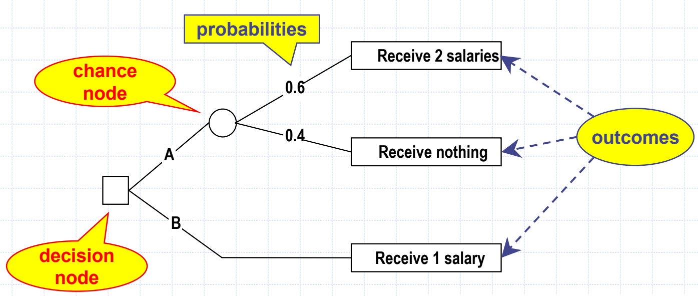
	Cost	
Alternatives	p=0.9	p=0.1
A	100	1000
B	150	550

- Multicriteria

	scenarios			
	C1		C2	
	cost	env	cost	env
X	1000	0.9	1000	0.8
Y	800	1.6	900	1.9
Z	500	1.7	1300	2
prob	0.7		0.3	

Decision trees - basics

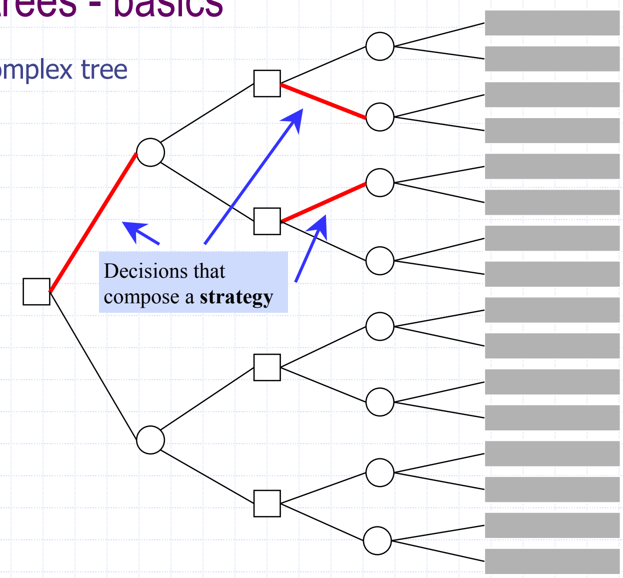
- A systematic way to represent a sequence of decisions and uncertain events through time, along with their outcomes
  - Complemented with procedures that identify the best strategy, according to some **decision paradigm**



Decision	1st Scen. (0.6)	2nd Scen.(0.4)
A	2 salaries	0
B	1 salary	1 salary

Decision trees - basics

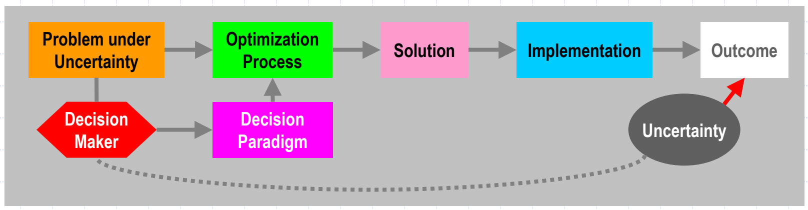
- A more complex tree



Applying decision paradigms

Decision paradigms

- Problems under uncertainty
  - Sometimes, the risk attitude of the DM is incorporated in the form of a pre-defined decision paradigm (expected value, regret, etc.)
  - This leads generally to an optimization process



Use of decision paradigms (or rules)

- Original problem
  - Dominated solutions shown

n	Cost C1 (0.3)	Cost C2 (0.6)	Cost C3 (0.1)
1	59	65	75
2	**50**	58	71
3	68	72	**60**
4	69	72	62
5	53	60	63
6	51	59	65
7	68	71	77
8	56	57	75
9	62	58	80
10	62	**55**	70

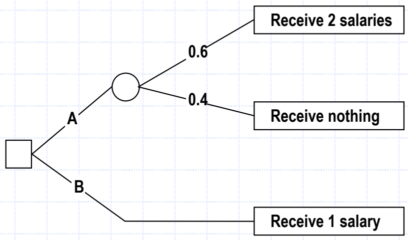
- Min E(Cost)

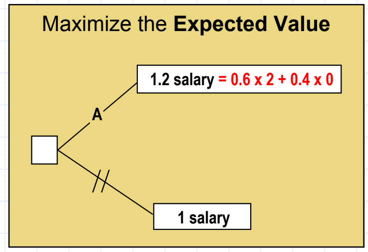
n	Expected Cost
1	64.2
2	**56.9**
3	69.6
4	70.1
5	58.2
6	57.2
7	70.7
8	58.5
9	61.4
10	58.6

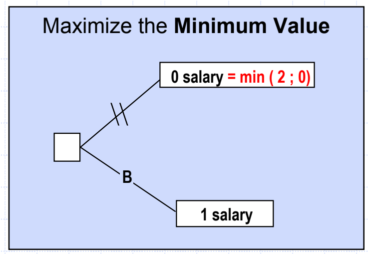
- Minimax Cost

n	Max Cost
1	75
2	71
3	72
4	72
5	**63**
6	65
7	77
8	75
9	80
10	70

Applying decision rules to a decision tree







- Procedure
  - (from the leaves to the root)
  - Chance nodes are collapsed according to the paradigm
  - Decision nodes are collapsed according to Max ou Min
  - Conditional decisions are saved

Decision paradigms for uncertainty

- Expected value paradigm
- Mean-variance (E-V) analysis
- Utility theory
- Robust optimization
- Bellman and Zadeh fuzzy decision

Expected value paradigm

- Basic paradigm
  - *Choose the alternative with the best expected value of the attribute*
  - *(implies risk indifference of the DM – linear utility)*

$$A_k = \{(z_{ik}, p_{ik}), i = 1..s\} \xrightarrow{EVP} A_k = \sum_{i=1}^{s} p_{ik} \cdot z_{ik}$$

$$A_k = f_k(z) \xrightarrow{EVP} A_k = \int_{-\infty}^{\infty} z.f_k(z)\,dz$$

Mean-variance (E-V) analysis

- Basic paradigm
  - *Choose the alternative that simultaneously has the best expected value of the attribute and the smaller variance*
    - *Greater variance for risk seeking DM!*
  - multicriteria approach (E-V diagrams are common)
- Value functions
  - Risk equivalents added to the expected value

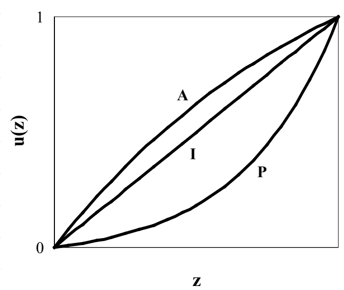
$$\max E[z] - \alpha.E\left[(z-\bar{z})^2\right]$$

$$\max E[z] - \alpha.E\left[(z-\bar{z})^2\right] + \beta.E\left[(z-\bar{z})^3\right]$$

α>0 for risk aversion, β>0 for skewness inclusion

Utility theory

- A formal way to include risk in the evaluation of alternatives
- Basic paradigm
  - Choose the alternative with the greatest expected utility
  - (incorporates the risk attitude of the DM through an utility function)
- Risk attitude:
  - Risk Aversion: concave function
  - Risk Proneness: convex function
  - Risk Indifference: linear function
    - Equivalent to the Expected Value P.



Utility functions – single attribute

- Decision rule: maximize the expected utility: $U(A) = \sum_{k=1}^{c} p_k U_k(A_k)$

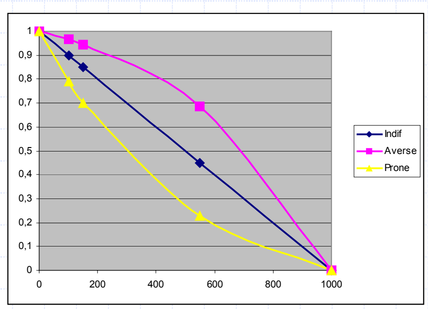
Cost			
Alternatives	p=0.9	p=0.1	E
A	100	1000	190
B	150	550	190

Utility Functions					
Cost	0	100	150	550	1000
Indif	1	0,90	0,85	0,45	0
Averse	1	0,97	0,95	0,69	0
Prone	1	0,79	0,70	0,23	0

Risk Indifferent			
Alternatives	p=0.9	p=0.1	U
A	0,90	0	**0,81**
B	0,85	0,45	**0,81**

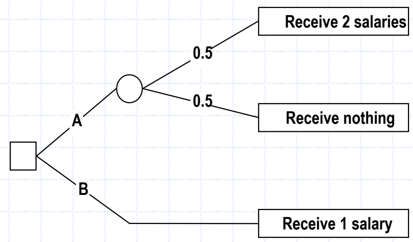
Risk Averse			
Alternatives	p=0.9	p=0.1	U
A	0,97	0	0,87
B	0,95	0,69	**0,92**

Risk Prone			
Alternatives	p=0.9	p=0.1	U
A	0,79	0	**0,71**
B	0,70	0,23	0,65



Utility functions – single attribute

- Check the risk profile of the DM
  - Prefers A? (Risk prone)
  - Prefers B? (Risk averse)
  - Indifferent? (Risk neutral)

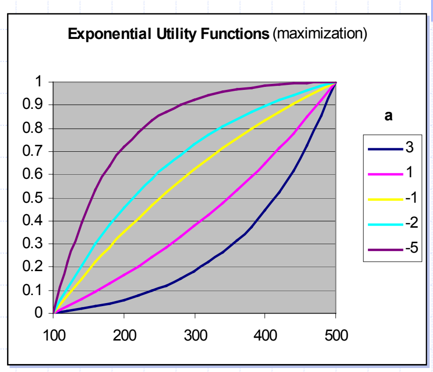


- Build the utility function
  - Using a predetermined form

    or
  - Point by point, using lotteries to interrogate the DM

Some useful functions...

- You may want to use a predefined function to ease your work
  - linear $U(X) = x = \frac{X - X^{worst}}{X^{best} - X^{worst}}$
  - polynomial $U(X) = x^k$
  - exponential $U(X) = \frac{e^{ax} - 1}{e^{a} - 1}$
    - a<0 means constant aversion
    - a>0 means constant proneness



Multiattribute utility functions

- When there are multiple criteria
- Typical forms:
  - additive $U(X)=\sum_{i=1}^{m} k_i U_i(X_i)$
  - multiplicative $1+kU(X)=\prod_{i=1}^{m}(k k_i U_i(X_i)+1)$
  - multilinear
    - 2 attributes $U(X)=k_1 U_1(X_1)+k_2 U_2(X_2)+k_{12} U_1(X_1) U_2(X_2)$

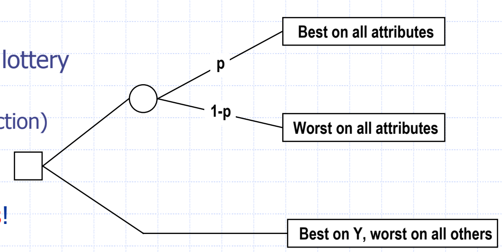
MA utility functions - parameters

- Build the multiattribute utility function
  - parameters result from judgments
    - Similar to value functions
    - eg two attributes

$$A \sim B \Rightarrow U(A)=U(B)$$

$$\begin{cases} k_1.U_1(a_1)+k_2.U_2(a_2)=k_1.U_1(b_1)+k_2.U_2(b_2) \\ k_1+k_2=1 \end{cases}$$

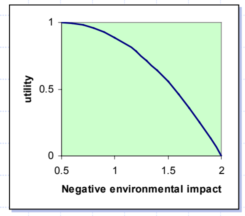
  - or decisions between a lottery and a sure value
    - eg k_Y=p (additive function)



  - but never from guesses!

MA utility function (example)

- X=(cost, negative environmental impact)
  - 500<cost<1500, 0.5<env<2.0



$$U_{cost}(cost) = \frac{1500 - cost}{1500 - 500}$$

$$U_{env}(env) = 1 - \frac{(env - 0.5)^2}{(2 - 0.5)^2}$$

  - A=(1000, 0.5) ~ B=(700, 1.7)

$$U_{cost}(A) = 0.5 \qquad U_{env}(A) = 1$$

$$U_{cost}(B) = 0.8 \qquad U_{env}(B) = 0.36$$

$$U(A) = U(B)$$

$$k_1.U_{cost}(A) + k_2.U_{env}(A) = k_1.U_{cost}(B) + k_2.U_{env}(B)$$

$$0.5\,k_1 + k_2 = 0.8\,k_1 + 0.36\,k_2$$

$$0.3k_1 - 0.64k_2 = 0$$

$$0.3k_1 - 0.64(1 - k_1) = 0$$

$$k_1 = 0.68 \qquad k_2 = 0.32$$

$$U(X) = 0.68\,U_{cost}(X) + 0.32\,U_{env}(X)$$

MA utility function (example)

- 3 alternatives X,Y,Z
  2 scenarios

	scenarios			
	C1		C2	
	cost	env	cost	env
X	1000	0.9	1000	0.8
Y	800	1.6	900	1.9
Z	500	1.7	1300	2
prob	0.7		0.3	

- Calculate the utilities in each scenario and the expected utility

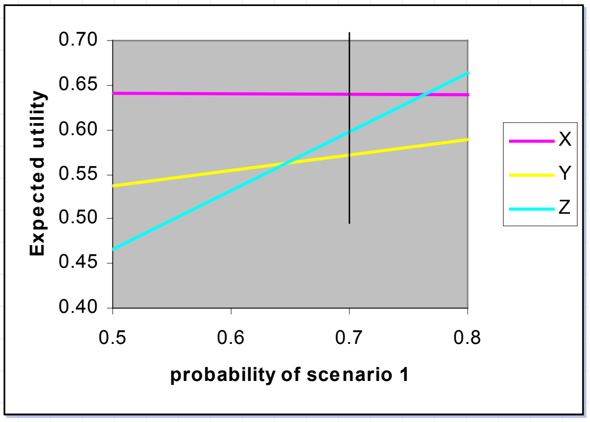
$$U(X) = 0.68\,U_{cost}(X) + 0.32\,U_{env}(X)$$

$$E(U(X)) = 0.7\,U_{sc1}(X) + 0.3\,U_{sc2}(X)$$

	scenarios						
	C1			C2			
	cost	env	**U1**	cost	env	**U2**	**E(U)**
X	0.5	0.93	**0.64**	0.5	0.96	**0.65**	**0.64**
Y	0.7	0.46	**0.62**	0.6	0.13	**0.45**	**0.57**
Z	1	0.36	**0.80**	0.2	0.00	**0.14**	**0.60**
prob	0.7			0.3			

Sensitivity analysis

- If your are not sure about the scenarios' probabilities you may want to study the sensitivity of the final order to them
  - example: p(C1) from 0.8 to 0.5



Robust optimization

- Idea
  - This approach deals with uncertainty by trying to avoid unpleasant outcomes in adverse scenarios
- Basic paradigm
  - *Choose the alternative that, in the worst case, has the best value (minimax paradigm)*
- Basic concepts
  - Robustness, disappointment, regret
- Single attribute approach
  - In multicriteria problems, you must first aggregate

Absolute robust approach

- When
  - goal satisfaction
  - situations where the uncertainty comes from competitors' decisions
- Rule
  - Choose the alternative corresponding to: $\min_{z \in Z} \max_{s \in S} Cost(z,c)$

Z - set of alternatives
S - set of scenarios

	Cost			
alternative	Sc 1	Sc 2	Sc 3	order
A	850	**1100**	900	2
B	500	**1650**	1600	4
C	**1200**	1100	1150	3
D	900	**1000**	950	1
E	500	800	**1700**	5

Minimax regret approach

- when the quality of the decision is evaluated *ex post facto*
- when your losses are automatically gains of your competitors
- Rule: $\min_{z \in Z} \max_{s \in S} Regret(z,s) = \min_{z \in Z} \max_{s \in S} \left(Cost(z,s) - Cost^*(s)\right)$

Best in scenario s

	Cost		
alternative	Sc 1	Sc 2	Sc 3
A	850	**1100**	900
B	500	**1650**	1600
C	**1200**	1100	1150
D	900	**1000**	950
E	500	800	**1700**
Best in Sc	500	800	900

	Regret			
alternative	Sc 1	Sc 2	Sc 3	order
A	**350**	300	0	1
B	0	**850**	700	5
C	**700**	300	250	3
D	**400**	200	50	2
E	0	0	**800**	4

Minimax *weighted* regret approach

- when scenarios have very different probabilities
- Rule: $\min_{z \in Z} \max_{s \in S} \{prob(s).Regret(z,s)\}$

	Regret		
alternative	Sc 1	Sc 2	Sc 3
A	**350**	300	0
B	0	**850**	700
C	**700**	300	250
D	**400**	200	50
E	0	0	**800**
prob	0.3	0.6	0.1

⇨

	Weighted Regret			
alternative	Sc 1	Sc 2	Sc 3	order
A	105	**180**	0	3
B	0	**510**	70	5
C	**210**	180	25	4
D	**120**	**120**	5	2
E	0	0	**80**	1

- We may have also uncertainty on the probabilities
  - e.g. modeled with intervals or fuzzy sets

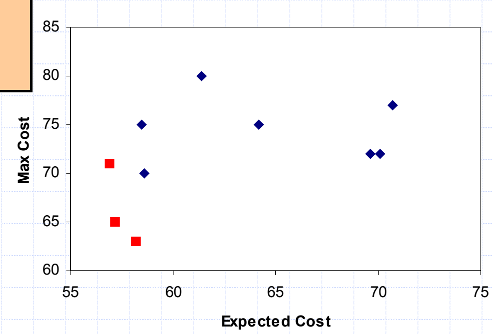
Using multiple (risk) indices

Using multiple indices

- Decision-aid in uncertain environments can be performed by constructing multicriteria models
  - Mathematically deterministic
- Traditional prescriptive approaches (decision paradigms) may be interpreted as possible points of view, leading to different attributes
  - The Uncertainty model and the Decision methodology are separated
- Other risk related attributes can be constructed and used
  - Must be meaningful for the Decision Maker
  - Problem dependent!

Multiple indices (example)

	Cost			Expected	Max
n	C1 (0.1)	C2 (0.6)	C3 (0.3)	Cost	Cost
1	59	65	75	64.2	75
2	50	58	71	**56.9**	**71**
3	68	72	60	69.6	72
4	69	72	62	70.1	72
5	53	60	63	**58.2**	**63**
6	51	59	65	**57.2**	**65**
7	68	71	77	70.7	77
8	56	57	75	58.5	75
9	62	58	80	61.4	80
10	62	55	70	58.6	70



Some traditional risk indices

- Used with probability models or scenarios
  - Variance (or Standard deviation)
  - Skewness
  - Probability of a negative outcome
  - Expected value of losses
  - Worst-case value
  - Regret
  - Exposure

$$Exposure(z) = \frac{\#\{s \in S \mid Regret(z,s) \leq threshold\}}{\#S}$$

Possible indices

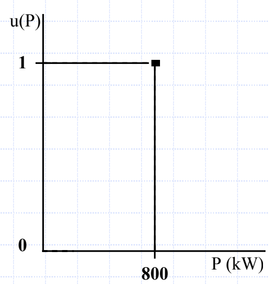
- Expected value (central measure attribute)
- Risk related indices
  - Variance, skewness, regret, etc.
  - Probability of an outcome worst than a specified value
  - Negative outcomes with a probability greater than a sp. Value
- "Optimistic" indices
  - Expected value of gains
  - Best-case value
  - Probability of a positive outcome
- Constraint related indices $p = p\left\{ \bigcup_{k \in C} \left(b_k(z) > b_k \right) \right\}$

Some ideas about fuzzy modeling

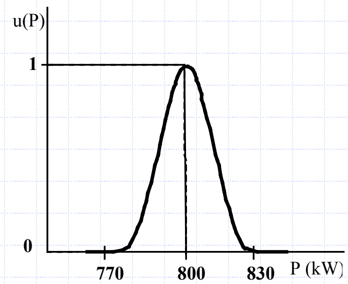
Why fuzzy sets?

- Fuzzy sets incorporate implicitly an infinite number of scenarios
  - But we can conjugate the two concepts
- Experts' knowledge can be "translated" in a simple way
- In most cases, uncertainty is better captured than with probabilities
  - Most of the times, no significant statistical data exists
  - But Fuzzy Sets are not a substitute for probabilistic models when these are adequate
- We can construct fuzzy-probabilistic methodologies
  - e.g. Reliability calculations when failure rates are fuzzy
- A bonus: Calculations are not difficult

Load fuzzy models



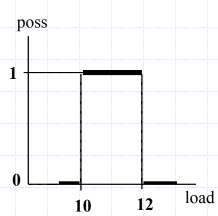
Crisp
"A load of 800 kW"



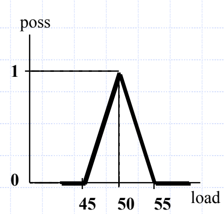
Fuzzy
"A load *about* 800 kW"

Load fuzzy models

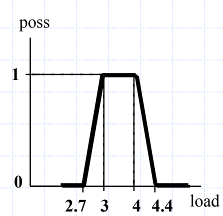
- A "dictionary" for qualitative declarations:



"Load between 10 and 12 MW"



"Load about 50 MW"



"Load more or less between 3 and 4 MW"

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With fuzzy data ...

- It is possible to develop physical and impact models that propagate this kind of uncertainty (extension principle)
- State variables may turn fuzzy
  - Fuzzy voltages (if loads are fuzzy)
  - Fuzzy branch currents (if loads are fuzzy)
- Attributes may turn fuzzy
  - Fuzzy Investment cost (if costs or rates are fuzzy)
  - Fuzzy Operation costs (if loads are fuzzy)
  - Fuzzy EENS (if loads or failure rates are fuzzy)
- New criteria may appear
  - Using central measures, risk indices and constraint violation indices

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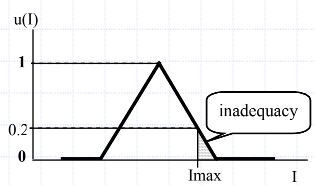
Indices for fuzzy models

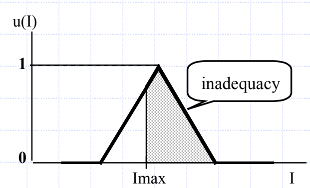
- Central measure attribute:
  - Removal
  - Center of gravity
  - Maximum value
- (additional) Risk related indices
  - Largest and smallest of maximum degree of membership
  - Largest and smallest possible values
  - Divergence
  - Set of possible results
  - Measures of fuzziness
    - a scalar index to measure the degree of fuzziness of a fuzzy set
- Constraint related indices
  - Robustness, severity, inadequacy

Constraint related indices

- Robustness
- Severity, Inadequacy



Small violation
Robustness = 0.8, Exposure = 0.2
Severity = Inadequacy/ Imax



Strong violation
Robustness = 0, Exposure = 1
Severity = Inadequacy/ Imax

Final remarks

- Decision problems result from the consideration of multiple criteria or because of uncertainty
  - The concept of optimum is no longer applicable
  - (But optimizing procedures are still needed!)
- The DM preferences must be incorporated in the process
  - The different methodologies try to help the DM doing so
  - Preferences are relative to criteria and/or risk
- Value functions result from a systematic building process
  - They are generally additive, but not necessarily linear
  - Parameters should be determined, not asked as naïve weights
- To deal with uncertainty, no decision paradigm prevails
  - Choosing one of them is a kind of meta-decision
  - We also may use multiple indices and transform the problem into a deterministic MC one