Value functions

- Allows us to construct an **order** of all the alternatives in a multicriteria **deterministic** environment
- eg

alternative	cost	λ	U
X	1000	0.10	5
Y	800	0.15	10
Z	500	0.21	11
	C$	/yr	h/yr

alternative	cost	risk
A	1000	5
B	800	10
C	500	12
D	850	10
E	1200	6

The first example is a typical situation in planning, when different investments correspond to different (estimated) quality of service, as measured by classical reliability indices.

Note that *risk*, in the second example, stands for a **deterministic measure of risk**, like, for instance, risk=prob*loss. Also, *cost* could be an **average cost**. This could be an alternative way of addressing an uncertainty situation (like E-V analysis in portfolio theory).

Of course, examples of deterministic problems could be presented, like choosing the best development project taking into account cost, environmental impact and amount of people served.

So, value functions can be used to deal with *true* deterministic problems (multicriteria problems) or with uncertainty situations where deterministic indices have previously been calculated.

Value functions - existence

- If Z is a subset of R^m
  - *i.e. if each alternative A is described by m attributes (A_1, A_2, ..., A_m)*

  and
  - ($A \geq B$ and $A \neq B$) $\Rightarrow A \succ B$ for all A, B $\in$ Z
  - For all A, B, C $\in$ Z such that $A \succ B \succ C$, it exists exactly one $\lambda \in (0, 1)$ such that $B \sim [\lambda.A + (1-\lambda).C]$

- Then, it exists a real **value function** *v()* such that:
  - $A \succ B \Leftrightarrow v(A) > v(B)$
  - $A \sim B \Leftrightarrow v(A) = v(B)$

Independence and additivity

- Given a set of attributes K, a subset X of K is said to be preferentially independent (p.i.) from its complement Y=K-X iff, for a particular value P_Y,

$$(A_X, P_Y) \succeq (B_X, P_Y) \Rightarrow (A_X, Q_Y) \succeq (B_X, Q_Y)$$

  - stands for all Q_Y, A and B being arbitrary.

- A set K is mutually preferentially independent (m.p.i.) if every subset X of K is p.i. from its complement K-X

- For three or more criteria (m>2), this is a sufficient condition to additivity:

$$A \succeq B \Rightarrow v_1(A_1) + \ldots + v_m(A_m) \geq v_1(B_1) + \ldots + v_m(B_m)$$

Additivity (m=2)

- For two criteria, an additional condition is necessary for additivity
  - For instance, the Thomsen condition:

$$\begin{array}{l} For\ \ all\ \ P;Q,A \\ (P_1,A_2)\sim(A_1,Q_2)\ and\ \ (A_1,P_2)\sim(Q_1,A_2)\Rightarrow(P_1,P_2)\sim(Q_1,Q_2) \end{array}$$

  - or the cancellation condition
    - *also guaranties that K is m.p.i.*

$$\begin{array}{l} For\ \ all\ \ P;Q,A \\ (P_1,A_2)\succeq(A_1,Q_2)\ and\ \ (A_1,P_2)\succeq(Q_1,A_2)\Rightarrow(P_1,P_2)\succeq(Q_1,Q_2) \end{array}$$

- More weak conditions exist for difficult cases

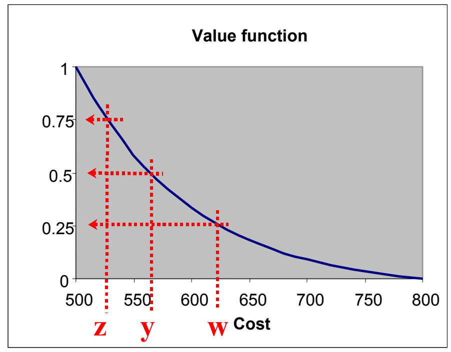
Building value functions

- Direct construction
  - Too complicated
- Verify preferential independence conditions
  - Then: $v(A) = \Psi(v_1(A_1), \ldots, v_m(A_m))$
- Check for additivity conditions...
  - If they hold: $v(A) = k_1 v_1(A_1) + k_2 v_2(A_2) + \ldots + k_m v_m(A_m)$
- ...or less restrictive conditions
  - That let you use (eg two normalized individual value functions)

$$v(A) = k_1 v_1(A_1) + k_2 v_2(A_2) + k_{12} v_1(A_1).v_2(A_2)$$

Building individual value functions

- Fix $v(x_{min})=1$, $v(x_{max})=0$
- Find y such that
  - $x_{max} \rightarrow y$ or $y \rightarrow x_{min}$
  - is indifferent to the DM
- Then, $v(y)=0.5$
- Repeat to find w (interval $[y, x_{max}]$)
  - *$v(w)=0.25$*
- and z (interval $[x_{min}, y]$)
  - *$v(z)=0.75$*
- ... (trace the curve)



- Verify! *The DM should be indifferent between $w \rightarrow y$ and $y \rightarrow z$*

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Note that this process differs from the utility theory case, since no lotteries are employed here (no uncertainty, no risk).

Different techniques can be used in order to obtain reference points (like y, z, w). For instance, the DM must be indifferent between getting two alternatives with cost y or one with cost 500 plus another with cost 800.

On the other hand, predefined functions may be used: linear, quadratic and exponential are again the most popular. In this approach, only one reference point is generally needed, but more can be used to estimate the parameters by regression.

MA value functions - parameters

- Assess the parameters $v(A) = k_1 v_1(A_1) + k_2 v_2(A_2)$
  - Build "extreme" alternatives:

Ideal : best A_1, best A_2	P : best A_1, worst A_2	Q : worst A_1, best A_2
$v = 1, v_1 = 1, v_2 = 1$	$v_1 = 1, v_2 = 0$	$v_1 = 0, v_2 = 1$
$k_1 + k_2 = 1$	$v(P) = k_1$	$v(Q) = k_2$

  - Ask for a judgement (eg: $P \succeq Q$, that implies $k_1 \geq k_2$)
  - Find $M = (z, \text{worst } A_2) \sim Q$
    - *Then:*

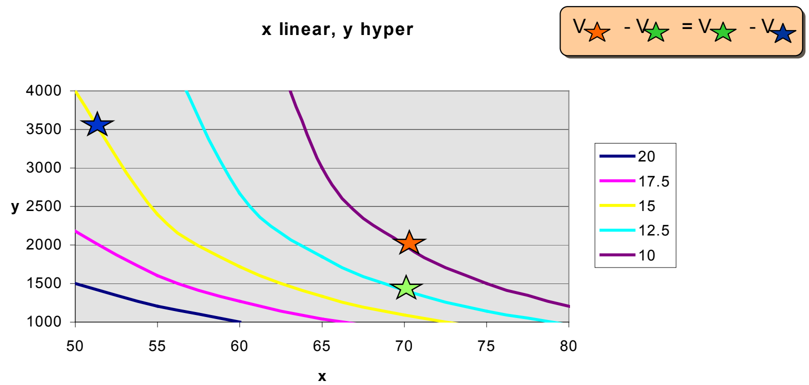
$$v(M) = v(Q) \Rightarrow k_1 v_1(z) = k_2$$

$$k_1 = \frac{1}{1 + v_1(z)} \qquad k_2 = 1 - k_1$$

    - *This is very different from asking directly for weights!*

An alternate look - indifference curves

- Indifference curves join all the points with the same value
- The DM is indifferent between two points in the same curve



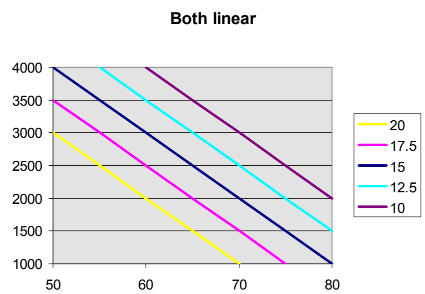
Note that, in the figure, values are not normalized (no problem with that, since the scale of a value function is only relative).

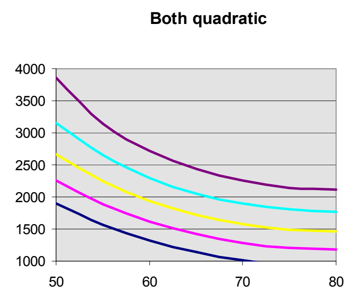
Indifference curves are similar to the lines we see in maps, corresponding to points with the same altitude. They show us indirectly the shape of the multiattribute value function. As they carry the same information, they can be used to find the preferred solution (the one that touches the curve with lesser value, in mimization problems).

Of course, graphical representation can only be used with two criteria

An alternate look - indifference curves

⌘ Other functions...





Conclusion - value functions

- A formal way to address multiattribute problems
- Requires
  - Verifying assumptions
  - Construction of the individual value functions
  - Indifference judgments
- Difficulties
  - Building individual value functions
- Problems
  - Tendency to use naïve weights asked directly to the DM