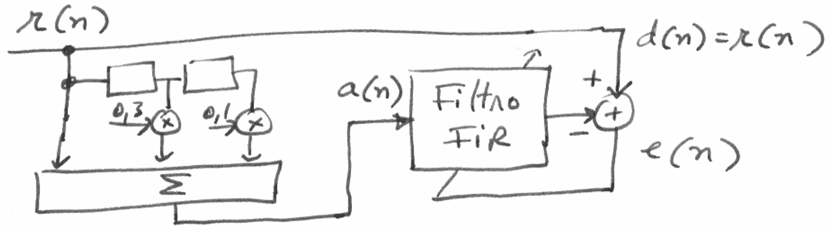
1) $a(n) = r(n) + 0{,}3\, r(n-1) + 0{,}1\, r(n-2)$



$E[d^2(n)] = E[r^2(n)] = 10$ ← σ^2

$R = E[\underline{a}(n)\,\underline{a}^T(n)]$, $\underline{p} = E[d(n)\,\underline{a}(n)]$

Seja R_k o termo genérico da 1.ª linha de $\underline{R}$.

$R_k = E[a(n)\,a(n-k)] = E\{[r(n)+0{,}3\,r(n-1)+0{,}1\,r(n-2)] \times$

$\times [r(n-k)+0{,}3\,r(n-k-1)+0{,}1\,r(n-k-2)]\} =$

$= 1{,}1\,\delta(k) + 0{,}33\,[\delta(k+1)+\delta(k-1)] + 0{,}1\,[\delta(k+2)+\delta(k-2)]$

(Considerou-se, para simplificar a expressão, que

$$E[r(n)\,r(n-k)] = \delta(k) = \begin{cases} \sigma^2 = 1/2 & k=0 \\ 0 & k \neq 0 \end{cases}$$, visto as

amostras de ruído branco serem descorrelacionadas entre si)

Da mesma forma, $\underline{p} = E[d(n)\,\underline{a}(n)] \Rightarrow p_k = E[d(n)\,a(n-k)]$

$\Rightarrow p_k = E[r(n)\,[r(n-k)+0{,}3\,r(n-k-1)+0{,}1\,r(n-k-2)]] =$

$= \delta(k) + 0{,}3\,\delta(k+1) + 0{,}1\,\delta(k+2)$

a) $N=5$ coeficientes

(c/ $\sigma^2 = 10$)

$$R = \sigma^2 \begin{bmatrix} 1{,}1 & 0{,}33 & 0{,}1 & 0 & 0 \\ 0{,}33 & 1{,}1 & 0{,}33 & 0{,}1 & 0 \\ 0{,}1 & 0{,}33 & 1{,}1 & 0{,}33 & 0{,}1 \\ 0 & 0{,}1 & 0{,}33 & 1{,}1 & 0{,}33 \\ 0 & 0 & 0{,}1 & 0{,}33 & 1{,}1 \end{bmatrix} = \begin{bmatrix} 11 & 3{,}3 & 1 & 0 & 0 \\ 3{,}3 & 11 & 3{,}3 & 1 & 0 \\ 1 & 3{,}3 & 11 & 3{,}3 & 1 \\ 0 & 1 & 33 & 11 & 3{,}3 \\ 0 & 0 & 1 & 33 & 11 \end{bmatrix}$$

$$\underline{p} = \sigma^2 \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \end{bmatrix}^T = \begin{bmatrix} 10 & 0 & 0 & 0 & 0 \end{bmatrix}^T$$

b) $N=2 \Longrightarrow \underline{R} = \sigma^2 \begin{bmatrix} 1,1 & 0,33 \\ 0,33 & 1,1 \end{bmatrix}$ e $p = \begin{bmatrix} \sigma^2 \\ 0 \end{bmatrix}$ ②

$$c_{opt} = R^{-1} p = \begin{bmatrix} 0,9990 \\ -0,2997 \end{bmatrix} \quad \text{(Valor independente de } \sigma^2\text{)}$$

$$\left(R^{-1} = \frac{1}{\sigma^2} \begin{bmatrix} 0,9990 & -0,2997 \\ -0,2997 & 0,999 \end{bmatrix} \right)$$

c) Do formulário : $\varepsilon = E[d^2(n)] - 2\underline{c}^T \underline{p} + \underline{c}^T R \underline{c}$

$$\varepsilon_{min} = E[d^2(n)] - p^T \underline{c}_{opt}$$

$$\underline{c}_{opt} = R^{-1} \underline{p} \Rightarrow \underline{p} = R \underline{c}_{opt}$$

Subtraindo ε_{min} a ε e desembaraçando de $\underline{p} = Rc_{opt}$ obtemos :

$$\varepsilon - \varepsilon_{min} = \underline{c}^T R \underline{c} - (2\underline{c}^T - c_{opt})^T p = \underline{c}^T R \underline{c} - c^T \underline{p} - (\underline{c} - \underline{c}_{opt})^T \underline{p} =$$

$$= \underline{c}^T R \underline{c} - \underline{c}^T R \underline{c}_{opt} - (\underline{c} - \underline{c}_{opt})^T R \underline{c}_{opt}$$

$$= \underbrace{c^T R (\underline{c} - \underline{c}_{opt})}_{(\underline{c} - \underline{c}_{opt})^T R c} - (\underline{c} - \underline{c}_{opt})^T R \underline{c}_{opt} =$$

$$= (\underline{c} - \underline{c}_{opt})^T R (\underline{c} - \underline{c}_{opt}) = \underline{c}_e^T R \underline{c}_e$$

Mas $R = Q \Lambda Q^T \Rightarrow \varepsilon - \varepsilon_{min} = \underline{c}_e^T Q \Lambda \underbrace{Q^T \underline{c}_e}_{\nu} =$

$$= \underbrace{(Q^T \underline{c}_e)^T}_{\nu^T} \Lambda \underline{\nu} = \underline{\nu}^T \Lambda \underline{\nu}$$

Ora $\underline{\nu} = \begin{bmatrix} \nu_0 \\ \nu_1 \end{bmatrix}$ e $\Lambda = \begin{bmatrix} \lambda_0 & 0 \\ 0 & \lambda_1 \end{bmatrix}$

$$\Rightarrow \varepsilon - \varepsilon_{min} = [\nu_0 \ \nu_1] \begin{bmatrix} \lambda_0 & 0 \\ 0 & \lambda_1 \end{bmatrix} \begin{bmatrix} \nu_0 \\ \nu_1 \end{bmatrix} = \lambda_0 \nu_0^2 + \lambda_1 \nu_1^2$$

As curvas de nível obtêm-se fazendo $\varepsilon - \varepsilon_{min} = \text{const.} = K$

$$\rightarrow \lambda_0 \nu_0^2 + \lambda_1 \nu_1^2 = K \Longrightarrow \frac{\nu_0^2}{K/\lambda_0} + \frac{\nu_1^2}{K/\lambda_1} = 1 \quad \text{c.q.d.}$$

Os semi-eixos valem : eixo dos xx : $a = \sqrt{K/\lambda_0} = \sqrt{\frac{\varepsilon - \varepsilon_{min}}{\lambda_0}}$

eixo dos yy : $b = \sqrt{K/\lambda_1} = \sqrt{\frac{\varepsilon - \varepsilon_{min}}{\lambda_1}}$

2 – $\Lambda = \begin{bmatrix} 1 & 0 \\ 0 & 5 \end{bmatrix}$ $Q = \frac{1}{\sqrt{2}} \begin{bmatrix} -1 & 1 \\ 1 & 1 \end{bmatrix}$ $\underline{p} = \begin{bmatrix} 5 \\ 3 \end{bmatrix}$

$\varepsilon_{min} = 0{,}6$

a) $R = Q \Lambda Q^T = \begin{bmatrix} 3 & 2 \\ 2 & 3 \end{bmatrix}$

Do formulário: $\varepsilon_{min} = E[d^2(n)] - p^T \underline{c}_{opt}$

$\Rightarrow E[d^2(n)] = \varepsilon_{min} + \underline{p}^T \underline{c}_{opt}$

Eq. Wiener-Hopf: $\underline{c}_{opt} = R^{-1} \underline{p} = \begin{bmatrix} 1{,}8 \\ -0{,}2 \end{bmatrix}$

Substituindo valores: $E[d^2(n)] = 0{,}6 + \underbrace{[5 \;\; 3] \begin{bmatrix} 1{,}8 \\ -0{,}2 \end{bmatrix}}_{8{,}4} = 9$

b) $\varepsilon = \varepsilon_{min} + \underline{c}_e^T R \underline{c}_e = 0{,}6 + 0{,}1 = 0{,}7$

(c/ $c_e = [0{,}1 \;\; 0{,}1]^T$)

c) Gama máxima: $0 < \mu < \frac{1}{\lambda_{max}} = \frac{1}{5} = 0{,}2$

Gama reduzida: $0 < \mu < \frac{1}{tr[R]} = \frac{1}{6}$

d) $\mu_{opt} = \frac{1}{\lambda_{min} + \lambda_{max}} = \frac{1}{1+5} = \frac{1}{6}$

e) O modo natural associado ao valor próprio mais elevado converge mais rapidamente (pois nesse caso $1-2\mu\lambda$ é menor). Portanto, sendo $\lambda_0 = 1$ e $\lambda_1 = 5$, o modo natural mais rápido é ν_1.

f) $\underline{c}_e(0) = [-1{,}8 \;\; 0{,}2]^T$. Sabemos que (ver alínea b)

$\varepsilon = \varepsilon_{min} + \underline{c}_e^T R \underline{c}_e = \varepsilon_{min} + c_e^T Q \Lambda Q^T \underline{c}_e = \varepsilon_{min} + \nu^T \Lambda \nu$

Explicitando a iteração $\underline{\underline{n}}$ temos

$\varepsilon(n) = \varepsilon_{min} + \nu^T(n) \Lambda \nu(n)$

Mas $\nu(n) = (I - 2\mu\Lambda)^n \nu(0)$

De $\nu(0) = Q^T \underline{c}_e(0)$ tiramos

$$\nu(0) = \frac{1}{\sqrt{2}} \begin{bmatrix} -1 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} -1,8 \\ 0,2 \end{bmatrix} = \frac{1}{\sqrt{2}} \begin{bmatrix} 2 \\ -1,6 \end{bmatrix}$$

Temos também $(I - 2\mu\Lambda)^n = \begin{bmatrix} (1-2\mu\lambda_0)^n & 0 \\ 0 & (1-2\mu\lambda_1)^n \end{bmatrix} =$

$$= \begin{bmatrix} 0,7^n & 0 \\ 0 & (-0,5)^n \end{bmatrix}$$

Substituindo valores e desenvolvendo:

$$\nu(n) = \frac{1}{\sqrt{2}} \left[2 \times 0,7^n \quad -1,6 \times (-0,5)^n \right]^T$$

$$\mathcal{E}(n) = \mathcal{E}_{min} + 2 \times 0,7^{2n} + 6,4 \times (0,5)^{2n}$$

Verificações:

$$\mathcal{E}(0) = \underbrace{\mathcal{E}_{min}}_{0,6} + 2 + 6,4 = 9$$

Note-se que se $\underline{c}_e(0) = [-1,8 \;\; 0,2]^T$ e $\underline{c}_{opt} = [1,8 \;\; -0,2]^T$ (alínea a)) então $\underline{c}(0) = [0 \;\; 0]^T$

$$\Rightarrow \mathcal{E}(0) = E[d^2(n)] - \underbrace{2\underline{c}^T p + \underline{c}^T R \underline{c}}_{0} =$$

$$= E[d^2(n)] = 9 \text{ (alínea a))}$$

$$\mathcal{E}(4) = \underbrace{\mathcal{E}_{min}}_{0,6} + \underbrace{2 \times 0,7^8 + 6,4 \times (0,5)^8}_{0,140} = 0,74$$ (tal como está no enunciado)

3- $J(n) = E[e^2(n)] + \gamma \underline{c}^T(n) \underline{c}(n)$

a) Equação de actualização: $\underline{c}(n+1) = \underline{c}(n) - \mu \hat{\nabla}(n)$

$$\nabla(n) = \frac{dJ(n)}{\underline{c}(n)} = 2E\left[e(n) \frac{de(n)}{d\underline{c}}\right] + 2\gamma \underline{c}(n)$$

Mas $e(n) = d(n) - \underline{c}^T(n) \underline{a}(n) \Rightarrow \dfrac{de(n)}{d\underline{c}} = -\underline{a}(n)$

$$\Rightarrow \nabla(n) = -2E[e(n)\underline{a}(n)] + 2\gamma \underline{c}(n)$$

Simplificando: $\hat{\nabla}(n) = -2e(n)\underline{a}(n) + 2\gamma \underline{c}(n)$

$$\Rightarrow \underline{c}(n+1) = \underline{c}(n) + 2\mu e(n)\underline{a}(n) - 2\mu\gamma \underline{c}(n) =$$

$$= (1 - 2\mu\gamma)\underline{c}(n) + 2\mu e(n)\underline{a}(n)$$ c.q.d.

3.b) Tomando o valor médio de ambos os membros da equação de actualização de coeficientes obtemos:

$$E[\underline{c}(n+1)] = (1-2\mu\gamma)\,E[\underline{c}(n)] + 2\mu\,E[e(n)\,\underline{a}(n)]$$

Ora $E[e(n)\,\underline{a}(n)] = E\Big[\big(d(n) - \underbrace{\underline{c}^T(n)\,\underline{a}(n)}_{\text{escalar } = \underline{a}^T(n)\,\underline{c}(n)}\big)\,\underline{a}(n)\Big] =$

$$= \underbrace{E[d(n)\,\underline{a}(n)]}_{\underline{p}} - E[\underline{a}(n)\cdot\underline{a}^T(n)\cdot\underline{c}(n)] =$$

$$= \underline{p} - \underbrace{E[\underline{a}(n)\,\underline{a}^T(n)]}_{R}\,E[\underline{c}(n)] = \underline{p} - R\,E[\underline{c}(n)]$$

Substituindo em cima

$$E[\underline{c}(n+1)] = (1-2\mu\gamma)\,E[\underline{c}(n)] + 2\mu\,\underline{p} - 2\mu R\,E[\underline{c}(n)] =$$

$$= [I - 2\mu(R+\gamma I)]\,E[\underline{c}(n)] + 2\mu\,\underline{p}$$

(No LMS era $E[\underline{c}(n+1)] = (I-2\mu R)\,E[\underline{c}(n)] + 2\mu\,\underline{p}$)

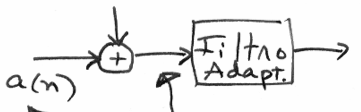
Quando $n \to \infty$: $E[\underline{c}(\infty)] \simeq [I - 2\mu(R+\gamma I)]\,E[\underline{c}(\infty)] + 2\mu\,\underline{p}$

$$\Rightarrow\ 2\mu(R+\gamma I)\,E[\underline{c}(\infty)] = 2\mu\,\underline{p}$$

$$\Rightarrow\ E[\underline{c}(\infty)] = (R+\gamma I)^{-1}\underline{p} \neq R^{-1}\underline{p} = \underline{c}_{opt}$$

c.q.d.

c) $\sigma_n^2 = \gamma \leftarrow R_n = \begin{bmatrix} \gamma & 0 & 0 \\ 0 & \gamma & \\ 0 & & \ddots\ \gamma \end{bmatrix} = \gamma I$



$R = E[\underline{a}(n)\,\underline{a}^T(n)]$

Como há independência entre $\underline{a}(n)$ e o "ruído" $\Rightarrow$ matriz de auto-correlação de entrada $= R + \gamma I$

Os valores próprios desta matriz são os valores próprios de R aumentados de $\gamma \Rightarrow \lambda_0+\gamma,\ \lambda_1+\gamma,\ \lambda_2+\gamma$, etc.

$$\Rightarrow\ 0 < \mu < \frac{1}{\lambda_{max}+\gamma}$$

4. Da figura vemos que

$\underline{a}(0)$	$\underline{a}(1)$	$\underline{a}(2)$	$\underline{a}(3)\ldots$
$\begin{bmatrix}1\\0\end{bmatrix}$	$\begin{bmatrix}-1\\1\end{bmatrix}$	$\begin{bmatrix}-1\\-1\end{bmatrix}$	$\begin{bmatrix}0\\-1\end{bmatrix}\ldots$

$d(0)$	$d(1)$	$d(2)$	$d(3)\ldots$
1	2	-2	0

$P(0) = 50I\ ,\quad \alpha = 1$

a) $$\hat{R}(n) = \sum_{j=0}^{n} \alpha^{n-j}\underline{a}(j)\underline{a}^T(j) = \underline{a}(0)\underline{a}^T(0) + \underline{a}(1)\underline{a}^T(1) + \ldots + \underline{a}(n)\underline{a}^T(n)$$

$$\hat{R}(1) = \underline{a}(0)\underline{a}^T(0) + \underline{a}(1)\underline{a}^T(1) = \begin{bmatrix}1\\0\end{bmatrix}\begin{bmatrix}1 & 0\end{bmatrix} + \begin{bmatrix}-1\\1\end{bmatrix}\begin{bmatrix}-1 & 1\end{bmatrix} =$$

$$= \begin{bmatrix}2 & -1\\-1 & 1\end{bmatrix}$$

b) $$\hat{p}(n) = \sum_{j=0}^{n}\alpha^{n-j}d(j)\underline{a}(j)$$

$$\hat{p}(2) = \sum_{j=0}^{2}\alpha^{2-j}d(j)\underline{a}(j) = d(0)\underline{a}(0) + d(1)\underline{a}(1) + d(2)\underline{a}(2) =$$

$$= \begin{bmatrix}1\\0\end{bmatrix} + 2\begin{bmatrix}-1\\1\end{bmatrix} + (-2)\begin{bmatrix}-1\\-1\end{bmatrix} = \begin{bmatrix}1\\4\end{bmatrix}$$

c) $$\underline{P}(n) = \frac{1}{\alpha}\left[\underline{P}(n-1) - \frac{\underline{P}(n-1)\,\underline{a}(n)\,\underline{a}^T(n)\,\underline{P}(n-1)}{\alpha + \underline{a}^T(n)\,\underline{P}(n-1)\,\underline{a}(n)}\right]$$

Sendo $\alpha = 1$ e $\underline{P}(0) = 50I$:

$$\underline{P}(1) = \underline{P}(0) - \frac{\underline{P}(0)\,\underline{a}(1)\,\underline{a}^T(1)\,\underline{P}(0)}{1 + \underline{a}^T(1)\,P(0)\,\underline{a}(1)}$$

Numerador:

$$50\begin{bmatrix}1 & 0\\0 & 1\end{bmatrix}\begin{bmatrix}-1\\1\end{bmatrix}\begin{bmatrix}-1 & 1\end{bmatrix}\times 50\begin{bmatrix}1 & 0\\0 & 1\end{bmatrix} =$$

$$= 2500\begin{bmatrix}1 & -1\\-1 & 1\end{bmatrix}\begin{bmatrix}1 & 0\\0 & 1\end{bmatrix} = 2500\begin{bmatrix}1 & -1\\-1 & 1\end{bmatrix}$$

Denominador:

$$1 + \begin{bmatrix}-1 & 1\end{bmatrix}\begin{bmatrix}1 & 0\\0 & 1\end{bmatrix}\begin{bmatrix}-1\\1\end{bmatrix}\times 50 = 1 + 2\times 50 = 101$$

$$\Rightarrow \underline{P}(1) = 50\begin{bmatrix}1 & 0\\0 & 1\end{bmatrix} - \underbrace{\frac{2500}{101}}_{\approx 24{,}75}\begin{bmatrix}1 & -1\\-1 & 1\end{bmatrix} = \begin{bmatrix}25{,}25 & 24{,}75\\24{,}75 & 25{,}25\end{bmatrix}$$

4.d) $\underline{k}(n) = \underline{P}(n)\,\underline{a}(n)$ — vector de ganho de Kalman

$$\Rightarrow \underline{k}(1) = \underline{P}(1)\,\underline{a}(1) = \begin{bmatrix} 25{,}25 & 24{,}75 \\ 24{,}75 & 25{,}25 \end{bmatrix} \begin{bmatrix} -1 \\ 1 \end{bmatrix} =$$

$$= \begin{bmatrix} -0{,}5 \\ 0{,}5 \end{bmatrix}$$

e) $e'(n) = d(n) - \underline{c}^T(n-1)\,\underline{a}(n)$ — erro de estimação "a priori"

$$\Rightarrow e'(1) = d(1) - \underline{c}^T(0)\,\underline{a}(1) =$$

$$= 2 - \begin{bmatrix} 0 & 0 \end{bmatrix} \begin{bmatrix} -1 \\ 1 \end{bmatrix} = 2$$

$$e'(2) = d(2) - \underline{c}^T(1)\,\underline{a}(2) = -2 - \begin{bmatrix} -1 & 1 \end{bmatrix} \begin{bmatrix} -1 \\ -1 \end{bmatrix} = -2$$

f) $\underline{c}(n) = \underline{c}(n-1) + \underline{k}(n)\,e'(n)$

$$\Rightarrow \underline{c}(1) = \underline{c}(0) + \underline{k}(1)\,e'(1) = \underline{0} + \begin{bmatrix} -0{,}5 \\ 0{,}5 \end{bmatrix} \times 2 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$