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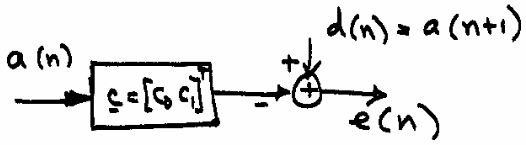
1º Mini-teste de CCD (04-05) — 9-11-04

1.



Este filtro é um preditor que tenta estimar o valor seguinte de $a(n)$, $d(n) = a(n+1)$.

$a(n) = w(n) + 0{,}4\, w(n-1) + 0{,}2\, w(n-2)$ $\quad w(n)$ — ruído branco c/ σ_w^2

a) $R = E\left[\underline{a}(n)\, \underline{a}^T(n)\right] = \begin{bmatrix} R_{11} & R_{12} \\ R_{21} & R_{22} \end{bmatrix}$, com $R_{12} = R_{21}$ e $R_{11} = R_{22}$.

$R_{11} = E\left[a^2(n)\right] = \cdots = (1 + 0{,}16 + 0{,}04)\sigma_w^2 = 1{,}2\,\sigma_w^2$

$R_{12} = E\left[a(n)\,a(n-1)\right] = \cdots = (0{,}4 + 0{,}08)\sigma_w^2 = 0{,}48\,\sigma_w^2$

(com $\sigma_w^2 = E\left[w^2(n)\right]$)

Do mesmo modo $p = \begin{bmatrix} p_{11} \\ p_{21} \end{bmatrix} = E\begin{bmatrix} d(n)\,a(n) \\ d(n)\,a(n-1) \end{bmatrix} = \begin{bmatrix} 0{,}48 \\ 0{,}2 \end{bmatrix}\sigma_w^2$

Coeficientes óptimos:

$$c_{opt} = R^{-1} p = \begin{bmatrix} 1{,}2\sigma_w^2 & 0{,}48\sigma_w^2 \\ 0{,}48\sigma_w^2 & 1{,}2\sigma_w^2 \end{bmatrix}^{-1} \begin{bmatrix} 0{,}48 \\ 0{,}2 \end{bmatrix}\sigma_w^2 = \begin{bmatrix} 0{,}3968 \\ 0{,}0079 \end{bmatrix}$$

(não depende de σ_w^2)

b) $$\varepsilon_{min} = E\left[d^2(n)\right] - p^T c_{opt} = \underbrace{E\left[a^2(n+1)\right]}_{1{,}2\sigma_w^2} - \underbrace{p^T c_{opt}}_{0{,}1921\sigma_w^2} = 1{,}0079\,\sigma_w^2$$

[Se $c_{opt} = \begin{bmatrix} 0{,}4 \\ 0 \end{bmatrix} \Rightarrow \varepsilon_{min} = (1{,}2 - 0{,}192)\sigma_w^2 = 1{,}008\,\sigma_w^2$]

2 $R = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$ $\quad p = [6 \;\; 4]^T$ $\quad \lambda_0 = 1 \Rightarrow \lambda_1 = 3$ pois $\sum_i \lambda_i = tr[R] = 4$.

$Q_0 = \frac{1}{\sqrt{2}}\begin{bmatrix} -1 \\ 1 \end{bmatrix}$. Sabemos que $Q_0^T Q_1 = 0$ e que $Q_1^T Q_1 = 1$ pois estamos a considerar vectores próprios normalizados.

$$Q_1 = \begin{bmatrix} q_{11} \\ q_{12} \end{bmatrix} \Rightarrow Q_0^T Q_1 = \frac{1}{\sqrt{2}} \begin{bmatrix} -1 & 1 \end{bmatrix} \begin{bmatrix} q_{11} \\ q_{12} \end{bmatrix} = 0$$

$$\Rightarrow -q_{11} + q_{12} = 0 \Rightarrow q_{11} = q_{12} > 0$$ (segundo o enunciado)

Mas $Q_1^T Q_1 = 1 \Rightarrow q_{11}^2 + q_{12}^2 = 2q_{11}^2 = 1 \Rightarrow q_{11}^2 = \frac{1}{2}$

$\Rightarrow q_{11} = \pm \frac{1}{\sqrt{2}}$. Só pode ser $q_{11} = \frac{1}{\sqrt{2}}$ pois tem de ser positivo.

Logo $Q_1 = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$, $Q = \frac{1}{\sqrt{2}} \begin{bmatrix} -1 & 1 \\ 1 & 1 \end{bmatrix}$ e $\Lambda = \begin{bmatrix} 1 & 0 \\ 0 & 3 \end{bmatrix}$.

a) $c_{opt} = R^{-1} p = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}^{-1} \begin{bmatrix} 6 \\ 4 \end{bmatrix} = \frac{2}{3} \begin{bmatrix} 4 \\ 1 \end{bmatrix} = \begin{bmatrix} 8/3 \\ 2/3 \end{bmatrix}$

b) $\varepsilon = \varepsilon_{min} + (\underline{c} - c_{opt})^T R (\underline{c} - c_{opt}) = \varepsilon_{min} + \underline{c}_e^T R \underline{c}_e =$

$= 5 + \begin{bmatrix} 1 & 1 \end{bmatrix} \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = 5 + 6 = 11$

d) $\underline{c}_e(0) = \begin{bmatrix} -2\sqrt{2} \\ -\frac{\sqrt{2}}{2} \end{bmatrix}$. Mas $\underline{v}(0) = Q^T \underline{c}_e(0) = \frac{1}{\sqrt{2}} \begin{bmatrix} -1 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} -2\sqrt{2} \\ -\frac{\sqrt{2}}{2} \end{bmatrix} = \begin{bmatrix} +\frac{3}{2} \\ -\frac{5}{2} \end{bmatrix}$

Não é preciso

Expressão analítica de ε (ou melhor, valor de ε) em $n=0$:

$\varepsilon(0) = \varepsilon_{min} + \underline{v}(0)^T \Lambda\, \underline{v}(0)$. No caso de dois coeficientes é

$\varepsilon(0) = \varepsilon_{min} + \lambda_0 v_0^2(0) + \lambda_1 v_1^2(0)$. Portanto, $\varepsilon(0) = 5 + 1 \times \frac{9}{4} + 3 \times \frac{25}{4} =$
$= 5 + 21 = 26$

No algoritmo do gradiente temos

$$\underline{v}(n) = (\underline{I} - 2\mu \Lambda)^n \underline{v}(0) = \begin{bmatrix} 1 - 2\mu\lambda_0 & 0 \\ 0 & 1 - 2\mu\lambda_1 \end{bmatrix}^n \begin{bmatrix} +3/2 \\ -5/2 \end{bmatrix}$$

Em $n = 4$: $\underline{v}(4) = \begin{bmatrix} v_0(4) \\ v_1(4) \end{bmatrix} = \begin{bmatrix} (1-2\mu\lambda_0)^4 \times (+3/2) \\ (1-2\mu\lambda_1)^4 \times (-\frac{5}{2}) \end{bmatrix} = \begin{bmatrix} +0{,}1944 \\ -0{,}004 \end{bmatrix}$

c) $D = \frac{\lambda_{max}}{\lambda_{min}} = \frac{3}{1} = 3$ e $\mu_{opt} = \frac{1}{\lambda_{min} + \lambda_{max}} = \frac{1}{4} = 0{,}25$

e) $\nabla(n) = -2p + 2Rc(n) \Rightarrow \underline{c}(n+1) = \underline{c}(n) - \mu \nabla(n) = (\underline{I} - 2\mu R)\, \underline{c}(n) + 2\mu p$

$$I - 2\mu R = \begin{bmatrix} 1-4\mu & -2\mu \\ -2\mu & 1-4\mu \end{bmatrix}$$

$$\underline{c}(1) = \begin{bmatrix} 1-4\mu & -2\mu \\ -2\mu & 1-4\mu \end{bmatrix} \begin{bmatrix} 0 \\ 0 \end{bmatrix} + 2\mu \begin{bmatrix} 6 \\ 4 \end{bmatrix} = \mu \begin{bmatrix} 12 \\ 8 \end{bmatrix}$$

$$\underline{c}(2) = \begin{bmatrix} 1-4\mu & -2\mu \\ -2\mu & 1-4\mu \end{bmatrix} \begin{bmatrix} 12\mu \\ 8\mu \end{bmatrix} + 2\mu \begin{bmatrix} 6 \\ 4 \end{bmatrix} = 4\mu \begin{bmatrix} 6-16\mu \\ 4-14\mu \end{bmatrix}$$

Substituindo $\mu = 0{,}2 \Longrightarrow \underline{c}(2) = \begin{bmatrix} 2{,}24 \\ 0{,}96 \end{bmatrix}$

Alternativa: usar $\underline{c}(n) - \underline{c}_{opt} = (I - 2\mu R)^n [\underline{c}(0) - \underline{c}_{opt}]$

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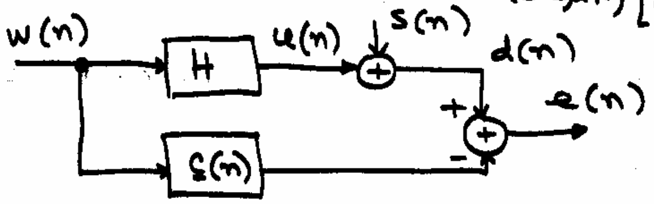

$\sigma_w^2 = \sigma_s^2 = 1$

Como $w(n)$ é ruído branco com $\sigma_w^2 = 1 \Longrightarrow R_w = \begin{bmatrix} \sigma_w^2 & 0 \\ 0 & \sigma_w^2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

a) Gama de valores de μ:

$0 < \mu < \dfrac{1}{\lambda_{max}}$ ou melhor ainda: $0 < \mu < \dfrac{1}{tr[R]} = \dfrac{1}{2}$

b)

$$\varepsilon_{min} = E[d^2(n)] - p^T c_{opt}$$

$$\underline{p} = E[d(n)\,\underline{w}(n)] = E[(u(n)+s(n))\,\underline{w}(n)] = E[u(n)\,\underline{w}(n)]$$

Ora $u(n) = [w(n) \quad w(n-1)] \begin{bmatrix} 1 \\ 3/2 \end{bmatrix} = w(n) + \frac{3}{2} w(n-1)$

$$\Longrightarrow \underline{p} = E \begin{bmatrix} w^2(n) + \frac{3}{2} w(n) w(n-1) \\ w(n) w(n-1) + \frac{3}{2} w^2(n-1) \end{bmatrix} = \begin{bmatrix} \sigma_w^2 \\ \frac{3}{2}\sigma_w^2 \end{bmatrix} = \begin{bmatrix} 1 \\ 3/2 \end{bmatrix}$$

$$\Longrightarrow c_{opt} = R^{-1} p = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 3/2 \end{bmatrix} = \begin{bmatrix} 1 \\ 3/2 \end{bmatrix} \quad \text{(claro!!)}$$

Como $E[d^2(n)] = E\{[s(n)+u(n)]^2\} = \cdots = \underbrace{E[u^2(n)]}_{\frac{13}{4}\sigma_w^2} + \underbrace{E[s^2(n)]}_{\sigma_s^2}$

e $\sigma_w^2 = \sigma_s^2 = 1 \Longrightarrow E[d^2(n)] = \frac{13}{4} + 1 = \frac{17}{4}$

Assim, $\varepsilon_{min} = \frac{17}{4} - [1 \quad 3/2] \begin{bmatrix} 1 \\ 3/2 \end{bmatrix} = \frac{17}{4} - \left(1 + \frac{9}{4}\right) = 1$

c) Desajuste: $\mathcal{M} = \frac{\varepsilon(\infty) - \varepsilon_{min}}{\varepsilon_{min}} \approx \mu \, tr[R] = 0{,}1$ (4)

$$\Rightarrow \quad \mu = \frac{0{,}1}{tr[R]} = \frac{0{,}1}{2} = 0{,}05$$

$$\varepsilon(\infty) = \varepsilon_{min} + M \varepsilon_{min} = (M+1)\,\varepsilon_{min} = 1{,}1\,\varepsilon_{min} = 1{,}1.$$

4 $P(0) = \hat{R}^{-1}(0) = 200\,I$, $\alpha = 1$

$a(n) = \cos \pi n$, $d(n) = \sqrt{2}\cos\pi(n - 1/4)$, $\underline{c}(0) = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$.

a) $$\underline{k}(n) = \frac{P(n-1)\,\underline{a}(n)}{\alpha + \underline{a}^T(n)\,P(n-1)\,\underline{a}(n)}$$

$n=1$ $$\underline{k}(1) = \frac{P(0)\,\underline{a}(1)}{\alpha + \underline{a}^T(1)\,P(0)\,\underline{a}(1)} = \frac{200\,\underline{a}(1)}{\underset{\uparrow \atop 1}{\alpha} + \underbrace{[-1 \quad 1] \cdot 200\,I \cdot \begin{bmatrix} -1 \\ 1 \end{bmatrix}}_{200\,(1+1) = 400}}$$

(pois $\underline{a}(1) = \begin{bmatrix} \cos\pi \\ \cos 0° \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$)

Logo, $$\underline{k}(1) = \frac{200\,[-1 \quad 1]^T}{401} = \frac{200}{401}\begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

b) $$e(n) = d(n) - c^T(n)\,\underline{a}(n) = d(n) - \left[\underline{c}(n-1) + \underline{k}(n)\,e'(n)\right]^T \underline{a}(n) =$$

$$= \underbrace{d(n) - \underline{c}^T(n-1)\,\underline{a}(n)}_{e'(n)} - e'(n)\,\underline{k}^T(n)\,\underline{a}(n) =$$

$$= \left[1 - \underline{k}^T(n)\,\underline{a}(n)\right] e'(n)$$

Se $\underline{k}(1) = \frac{200}{401}\begin{bmatrix} -1 \\ 1 \end{bmatrix}$ e como $e'(1) = d(1) - \underbrace{\underline{c}^T(0)}_{0}\,\underline{a}(1) = d(1) \Rightarrow$

$\Rightarrow e'(1) = d(1) = \sqrt{2}\cos\underbrace{\pi\left(1 - 1/4\right)}_{3\pi/4} = -1$ então

$$e(1) = \left(1 - \frac{200}{401}[-1 \quad 1]\begin{bmatrix} -1 \\ 1 \end{bmatrix}\right) e'(1) = -\frac{1}{401}$$